Quiz 8, MATH 246, Professor Radu Balan, Tuesday, 10 November 2009

Your Name: ____

UMD SID:____

Discussion Instructor (circle one): Jeremy Schwartz Geetanjali Kachari

Discussion Time (circle one): 8:00 9:00 10:00 11:00

1. [3pts] Use the definition of Laplace transform to compute the Laplace transform of

$$f(t) = \begin{cases} 1 & t \geq 6 \\ 0 & t < 6 \end{cases}$$

Solution

$$F(s) = \int_0^\infty e^{-ts} f(t)dt = \int_6^\infty e^{-ts} dt = -\frac{1}{s}\lim_{A\to\infty}\left[e^{-ts}\right]\Big|_6^A = -\frac{1}{s}\left[\lim_{A\to\infty} e^{-sA} - e^{-6s}\right] = \frac{e^{-6s}}{s}, s > 0$$

2. [7pts] Find the solution of the initial value problem:

$$\begin{cases} y''+y=g(t) \\ y(0)=0, y'(0)=1 \end{cases} \quad g(t)=\begin{cases} t/2 & 0\le t<6 \\ 3 & t\ge 6 \end{cases}$$

Short table: $\mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}}, \mathcal{L}\{\sin(at)\}=\frac{a}{s^2+a^2}, \mathcal{L}\{\cos(at)\}=\frac{s}{s^2+a^2}$

Solution

Let *Y(s)* and *G(s)* denote the Laplace transforms of *y(t)* and, respectively, *g(t)*.
The Laplace transform of the equation is:

$$s^2Y(s)-1+Y(s)=G(s)$$

Hence the Laplace transform of the solution is:

$$Y(s)=\frac{1+G(s)}{s^2+1}$$

Since

$$g(t)=\frac{t}{2}u(t)+\left(3-\frac{t}{2}\right)u(t-6)=\frac{t}{2}u(t)-\frac{t-6}{2}u(t-6)$$

It follows:

$$G(s)=\frac{1}{2s^2}-\frac{e^{-6s}}{2s^2}$$

And hence:

$$Y(s)=\frac{1-e^{-6s}}{2s^2\left(s^2+1\right)}+\frac{1}{s^2+1}$$

Expand in partial fractions:

$$\frac{1-e^{-6s}}{2}\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)=\frac{1}{2}\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)-\frac{1}{2}\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)e^{-6s}$$

Note: $\mathcal{L}^{-1}\left\{\frac{1}{s^2}-\frac{1}{s^2+1}\right\}=t-\sin(t)$ and thus:

$$y(t)=\frac{1}{2}\left(t-\sin(t)\right)-\frac{1}{2}\left(t-6-\sin(t-6)\right)u(t-6)+\sin(t)=$$

$$=\frac{1}{2}\left(t+\sin(t)\right)-\frac{1}{2}\left(t-6-\sin(t-6)\right)u(t-6)=\begin{cases} \frac{1}{2}\left(t+\sin(t)\right) & 0\le t<6 \\ 3+\frac{\sin(t)+\sin(t-6)}{2} & 6\le t \end{cases}$$

or:

$$y(t)=\begin{cases} \frac{1}{2}\left(t+\sin(t)\right) & 0\le t<6 \\ 3+2\cos(3)\sin(t-3) & 6\le t \end{cases}$$