Chapter 9 : Nonlinear Differential Equations and Stability

§9.1 : The Phase Plane for Linear Systems

$$\frac{d\underline{x}}{dt} = A\,\underline{x} \quad , \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Equilibrium points:

Want constant solutions: $\underline{x}(t) = \underline{x}_0$, all t.

$$\frac{d\underline{x}}{dt} \overset{!}{=} 0 = A\,\underline{x}(t) = A\,\underline{x}_0$$

Thus $\underline{x}_0 \in N(A) = \{ \underline{z} \in \mathbb{R}^2 \text{ such that } A\underline{z} = 0 \}$ (kernel, or null space of A)

$$P_A(s) = \det(A - sI) = \begin{vmatrix} a-s & b \\ c & d-s \end{vmatrix} = s^2 - (a+d)s + ad - bc = s^2 - tr(A)\cdot s + det(A)$$

$tr(A) = a+d$, $det(A) = ad - bc$.

Case 1 : $\det A = ad - bc \neq 0$.

Then the only constant solution is $\underline{x}(t) = 0$.

Case 2 : $\det A = 0$. : $\boxed{\text{Non isolated critical points}}$

Then there are ∞-many distinct constant solutions.

$$A\underline{v} = 0.$$

admits a nonzero solution $\underline{v}$. What about a second solution?

$\boxed{2.1}$ If there are two independent $\underline{v}^1, \underline{v}^2$ such that $A\underline{v}^1 = A\underline{v}^2 = 0$

then $A = 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Phase space:

and $\underline{x}(t) = \underline{x}(0)$, all t, all $\underline{x}(0)$

Every point is an equilibrium point.

$tr(A) = 0$
$det(A) = 0$

$$N(A) = \mathbb{R}^2.$$

$\boxed{2.2}$ If there is only one independant eigenvector associated to 0, for A,

$\underline{v}: \ A\underline{v} = 0$.

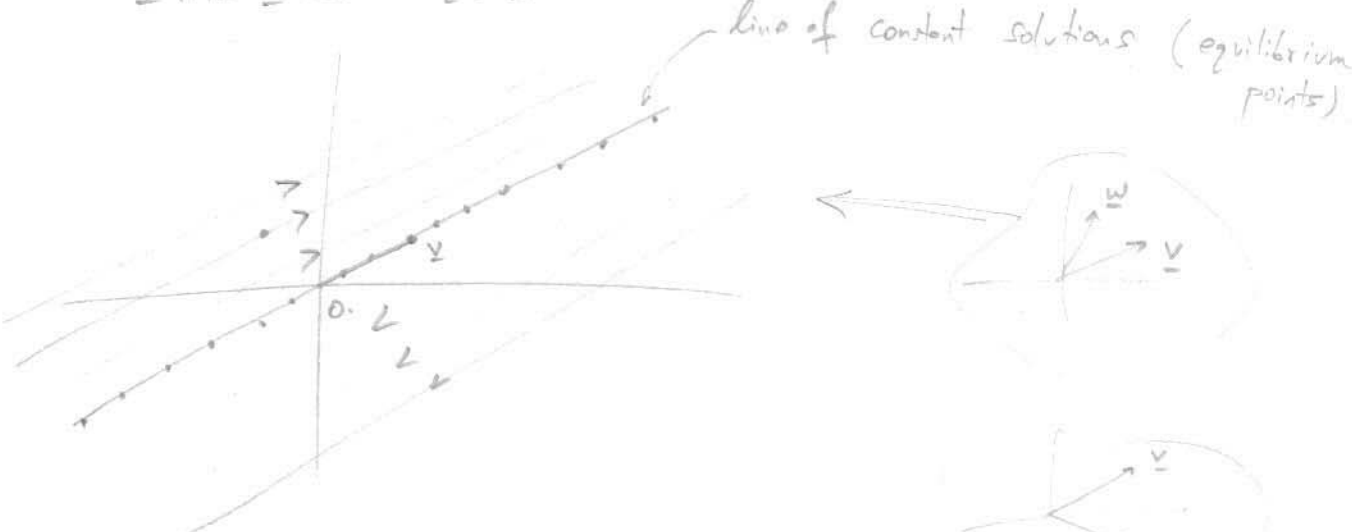
$\boxed{2.2.1}$ If $tr(A) = 0$, $p_A(s) = s^2 \rightarrow 0$ is a repeated eigenvalue.

Use generalized eigenvector theory: $\underline{x}^{(2)}(t) = t\,\underline{v} + \underline{w}$.

$$\left.\begin{aligned} &\frac{d}{dt}\underline{x} = \underline{v} \\ &A\underline{x} = t\underbrace{A\underline{v}}_{0} + A\underline{w} \end{aligned}\right\} \quad A\underline{w} = \underline{v} \ \rightarrow \text{solve for } \underline{w}.$$

General solution. $\underline{x}(t) = c_1\underline{v} + c_2(t\underline{v} + \underline{w}) = \underbrace{c_1\underline{v} + c_2\underline{w}}_{\underline{x}(0)} + c_2 t\underline{v}$

$\underline{x}(t) = \underline{x}(0) + c_2 t\underline{v}$



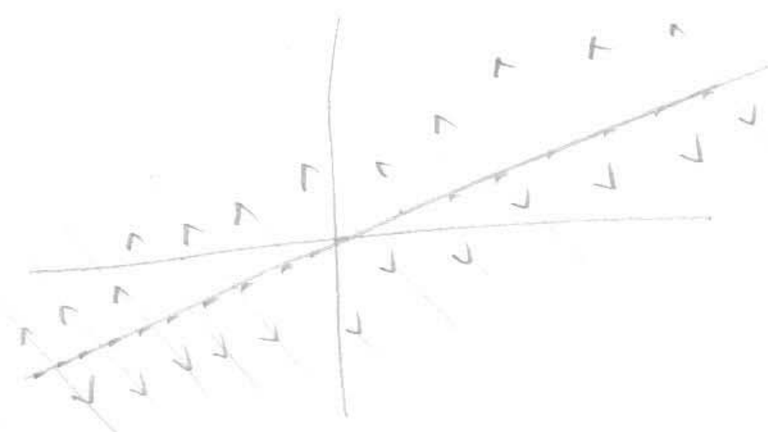
$\boxed{2.2.2.}$ $tr(A) \neq 0$: $p_A(s) = s^2 - tr(A)\, s = s\,[s - tr(A)]$

$$\rightarrow \quad s_1 = 0, \quad s_2 = tr(A) \neq 0$$

We have a second eigenvector: $A\,\underline{v}^2 = s_2 \cdot \underline{v}^2$

General solution: $\underline{x}(t) = c_1\,\underline{v} + c_2\,\underline{v}^2\, e^{tr(A)\cdot t}$

$tr(A) > 0$



Origin is unstable

$tr(A) < 0$

Origin is stable but not asymptotically stable

Case 1 : $\det(A) = ad - bc \neq 0$.

$$p_A(s) = s^2 - tr(A)\cdot s + \det(A) = 0 \implies s_1, s_2 \quad \text{and } s_1 \neq 0,\ s_2 \neq 0.$$

$$s_{1,2} = \frac{tr(A) \pm \sqrt{(tr(A))^2 - 4\det(A)}}{2}$$

$$\boxed{\begin{array}{l} s_1 + s_2 = tr(A) \\ s_1 \cdot s_2 = \det(A) \end{array}}$$

distinct eigenvalues : $(tr(A))^2 - 4\det(A) \neq 0$.

repeated eigenvalues : $(tr(A))^2 - 4\det(A) = 0$.

1.1 Distinct Eigenvalues : $s_1 \neq s_2$.

1.1.1 Real & distinct eigenvalues : $\Delta = (tr(A))^2 - 4\det(A) > 0$.

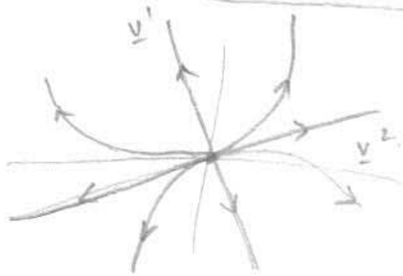
$$\det(A) < \tfrac{1}{4}(trA)^2.$$

$$\Rightarrow \underline{x}(t) = c_1 e^{s_1 t}\underline{v}^1 + c_2 e^{s_2 t}\underline{v}^2 \quad : \text{where} \begin{cases} A\underline{v}^1 = s_1\underline{v}^1 \\ A\underline{v}^2 = s_2 \underline{v}^2 \end{cases}$$

a). $s_1, s_2 > 0$: UNSTABLE NODE

Say $s_1 > s_2 > 0$

faster mode (s_1), slow mode (s_2).



$tr(A) > 0$, $\det(A) > 0$

b) $s_1, s_2 < 0$ say $s_1 < s_2 < 0$. Asymptotically stable NODE

$tr(A) < 0$, $\det(A) > 0$

c) $s_1 < 0 < s_2$: SADDLE

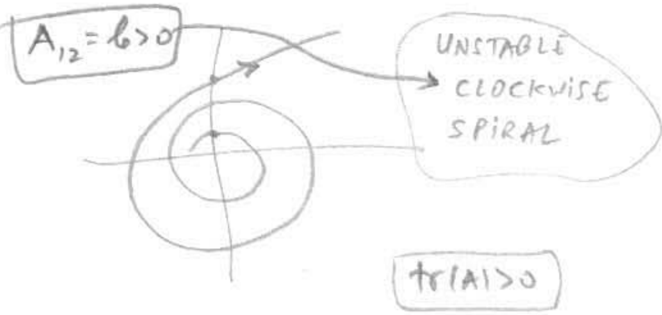
$\det(A) < 0$ ($tr(A)$ could be $> = < 0$)

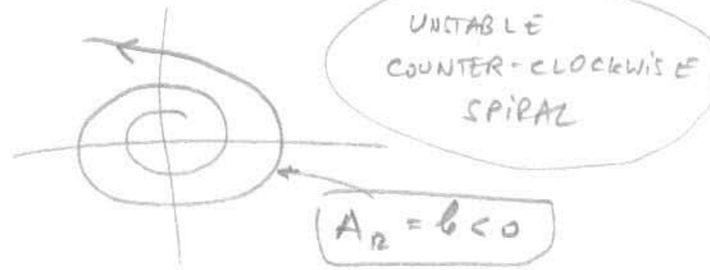
$\boxed{1.1.2}$ Complex eigenvalues.

$$S_{1,2} = \lambda \pm i\mu . \quad \Rightarrow \quad \underline{x}(t) = c_1 e^{\lambda t}\, Re\left[e^{i\mu}\,\underline{v}'\right] + c_2 e^{\lambda t}\, Im\left[e^{i\mu}\,\underline{v}'\right].$$

$2\lambda = tr(A)$, $\lambda^2 + \mu^2 = det(A)$

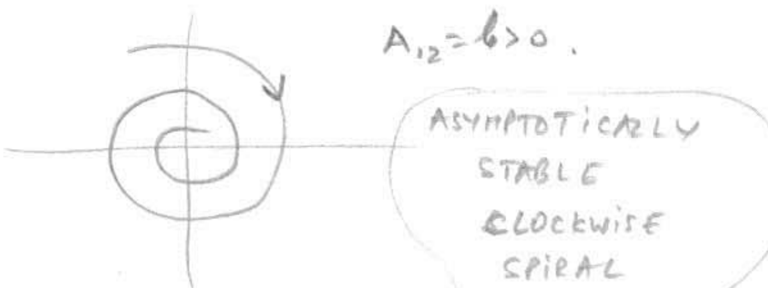
a) $\lambda > 0$: $\boxed{\text{UNSTABLE SPIRAL}}$



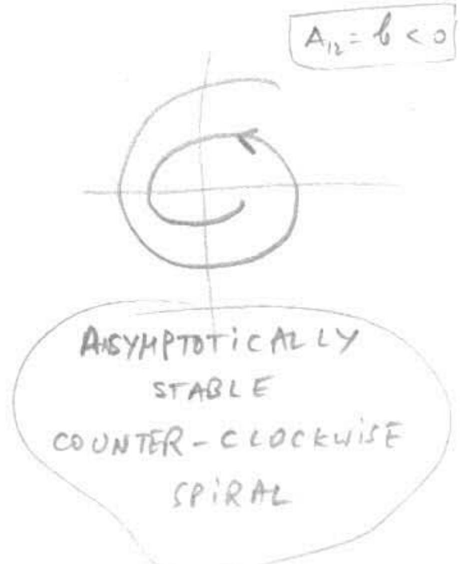


$\boxed{tr(A) > 0}$

b) $\lambda < 0$: $\boxed{\text{ASYMPTOTICALLY STABLE SPIRAL}}$



or



$\boxed{2\lambda = tr(A) < 0}$

c) $\lambda = 0$: $\quad tr(A) = 0, \quad det(A) = \mu^2 > 0$

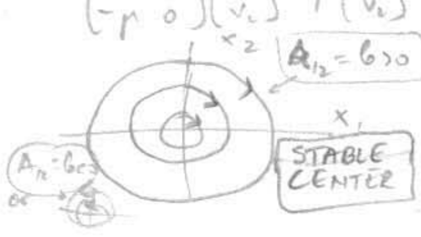
then: $\underline{x}(t) = c_1\, Re\left[e^{i\mu t}\underline{v}\right] + c_2\, Im\left[e^{i\mu t}\underline{v}\right]$

Typically: $A = \begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix} \rightsquigarrow$ eigenvector: $\begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = i\mu \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $\quad \mu v_2 = i\mu v_1 ; \quad v_2 = i v_1$

$\underline{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\underline{x}^1(t) = Re\left[e^{i\mu t}\begin{bmatrix} 1 \\ i \end{bmatrix}\right] = \begin{bmatrix} \cos(\mu t) \\ -\sin(\mu t) \end{bmatrix}$

$\underline{x}^2(t) = Im\left[e^{i\mu t}\begin{bmatrix} 1 \\ i \end{bmatrix}\right] = \begin{bmatrix} \sin(\mu t) \\ \cos(\mu t) \end{bmatrix}$



$\underline{x}(t) = \begin{bmatrix} c_1 \cos(\mu t) + c_2 \sin(\mu t) \\ -c_1 \sin(\mu t) + c_2 \cos(\mu t) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$\alpha^2 + \beta^2 = c_1^2 + c_2^2$

$\boxed{1.2}$ Repeated Eigenvalues $s_1 = s_2 \neq 0$.

$$tr(A) = 2s_1.$$

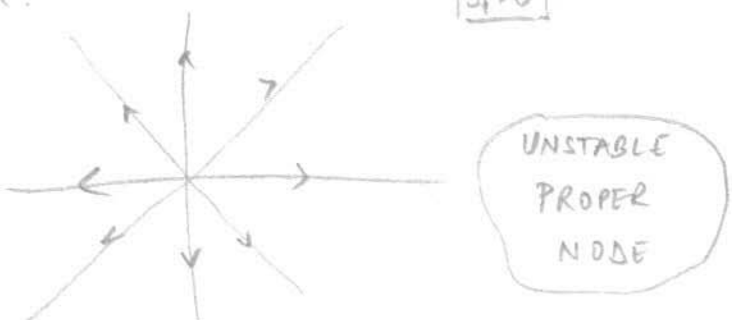
1.2.1: If there are two independent eigenvectors $\underline{v}^1, \underline{v}^2$

then $\underline{x}(t) = c_1 e^{s_1 t}\,\underline{v}^1 + c_2 e^{s_1 t}\,\underline{v}^2 = e^{s_1 t}\underbrace{\left[c_1\underline{v}^1 + c_2\underline{v}^2\right]}_{\underline{x}(0)} = e^{s_1 t}\,\underline{x}(0)$

Thus $A = s_1 I$ (must be!)

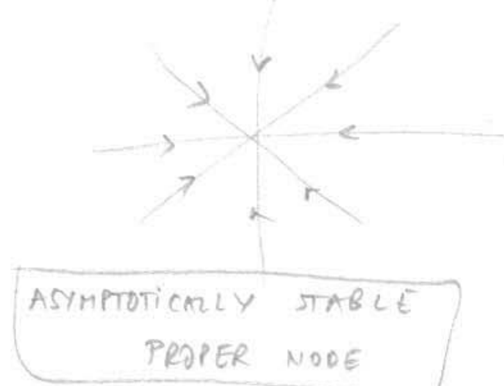
$$= \begin{pmatrix} s_1 & 0 \\ 0 & s_1 \end{pmatrix}$$

and $\boxed{s_1 > 0}$



$tr(A) > 0$.

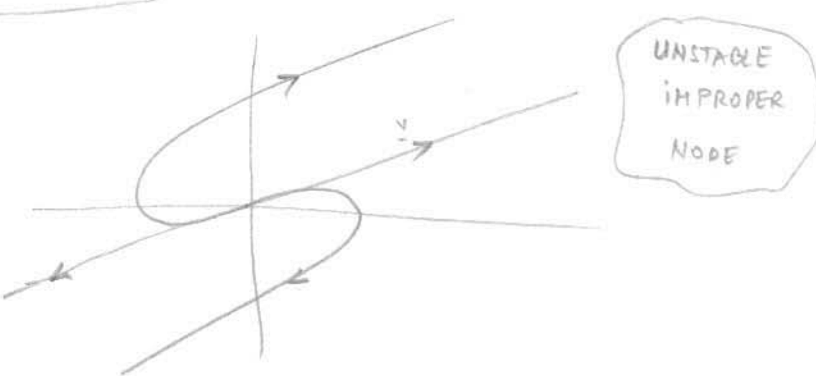
or $\boxed{s_1 < 0}$



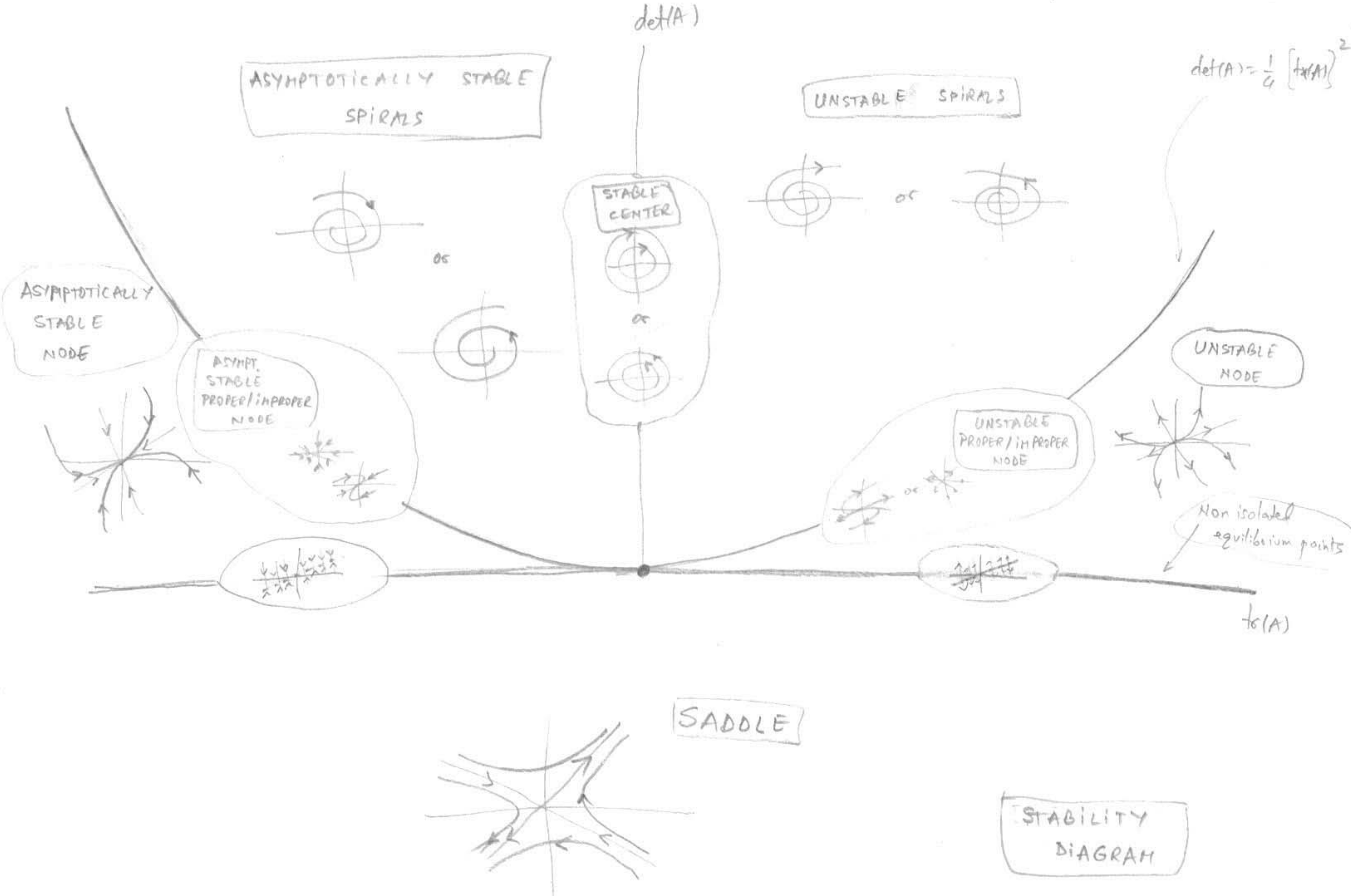
$tr(A) < 0$.

1.2.2: If there only one independent eigenvector $\underline{v}$, then the Second solution is $\underline{x}^2(t) = t\,e^{s_1 t}\cdot\underline{v} + e^{s_1 t}\,\underline{w}$

$\boxed{s_1 > 0 \cdot tr(A) > 0}$



$\boxed{s_1 < 0,\ tr(A) < 0}$


det(A)
ASYMPTOTICALLY STABLE SPIRALS
UNSTABLE SPIRALS
$\det(A) = \frac{1}{4}\left[\mathrm{tr}(A)\right]^2$
or
STABLE CENTER
or
or
ASYMPTOTICALLY STABLE NODE
ASYMPT. STABLE PROPER/IMPROPER NODE
UNSTABLE NODE
UNSTABLE PROPER/IMPROPER NODE
or
Non isolated equilibrium points
tr(A)
SADDLE
STABILITY DIAGRAM

§9.2 Autonomous Systems and Stability

Consider $\frac{d\underline{x}}{dt} = f(\underline{x})$, $\underline{x} \in D \subset \mathbb{R}^2$.

Definition:
Equilibrium points, or equilibrium solutions, or critical points are those points $\underline{x}^0$ such that $f(\underline{x}^0) = \underline{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Assume from now on that each critical point is isolated:
If $\underline{x}^0$ is a critical point, then there exists $r>0$ such that any other distinct critical point $\underline{y}^0$, $\|\underline{x}^0 - \underline{y}^0\| \geq r$.

Here: $\|\underline{z}\| = \sqrt{z_1^2 + z_2^2}$, where $\underline{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

Types of equilibrium: STABLE, ASYMPTOTICALLY STABLE, UNSTABLE.

Denote $\phi(t, \underline{y})$ the unique solution of the IVP $\begin{cases} \frac{d\underline{x}}{dt} = f(\underline{x}) \\ \underline{x}(0) = \underline{y} \end{cases}$ $\begin{cases} \frac{\partial \phi(t,y)}{\partial t} = f(\phi(t,\underline{y})) \\ \phi(0,y) = \underline{y} \end{cases}$

Definitions:

$\underline{x}^0$ stable: $\forall \varepsilon > 0\ \exists \delta > 0$, If $\underline{y}$, $\|\underline{x}^0 - \underline{y}\| < \delta$, then $\|\phi(t,\underline{y}) - \underline{x}^0\| < \varepsilon$ for all $t > 0$.

$\underline{x}^0$ asymptotically stable
i) it is stable
ii) there is $r > 0$, if $\underline{y}$, $\|\underline{x}^0 - \underline{y}\| < r$ then $\lim_{t \to \infty} \phi(t, \underline{y}) = \underline{x}^0$

$\underline{x}^0$ unstable: If it is not stable.

$\exists \varepsilon > 0\ \forall \delta > 0\ \exists \underline{y}$, $\|\underline{y} - \underline{x}^0\| < \delta$ and $\exists t > 0$, $\|\phi(t,\underline{y}) - \underline{x}^0\| \geq \varepsilon$.
$\underline{y} \neq \underline{x}^0$

§ 9.3 Linearization, or Locally Linear Systems

Consider $\dfrac{d\underline{x}}{dt} = f(\underline{x})$ (*)

and $\underline{x}^0$ an isolated critical point : $f(\underline{x}^0) = 0$.

Question : What can we say about stability of this equilibrium solution?

$$\underline{f}(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \quad ; \quad \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}.$$

Procedure :

1). Compute the Jacobian matrix :

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\underline{x}^0) & \frac{\partial f_1}{\partial x_2}(\underline{x}^0) \\ \frac{\partial f_2}{\partial x_1}(\underline{x}^0) & \frac{\partial f_2}{\partial x_2}(\underline{x}^0) \end{bmatrix}$$

NOTE: This is a 2×2 matrix of numbers

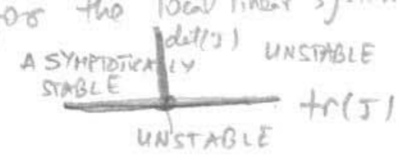
2). Analyze stability of origin for the system :

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = J \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad ; \quad \underline{u}' = J \cdot \underline{u}$$

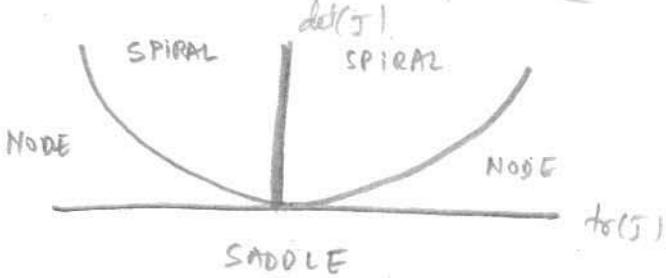
called the "local linear system".

3) If $\left(tr(J) = 0 \text{ and } det(J) < 0\right)$ or $\left(\begin{array}{c} det(J) \neq 0 \\ \text{and } tr(J) \neq 0 \end{array}\right)$ then the stability of the origin for the local linear system coincides with the stability of x^0 for (*).

asymptotically stable / unstable.

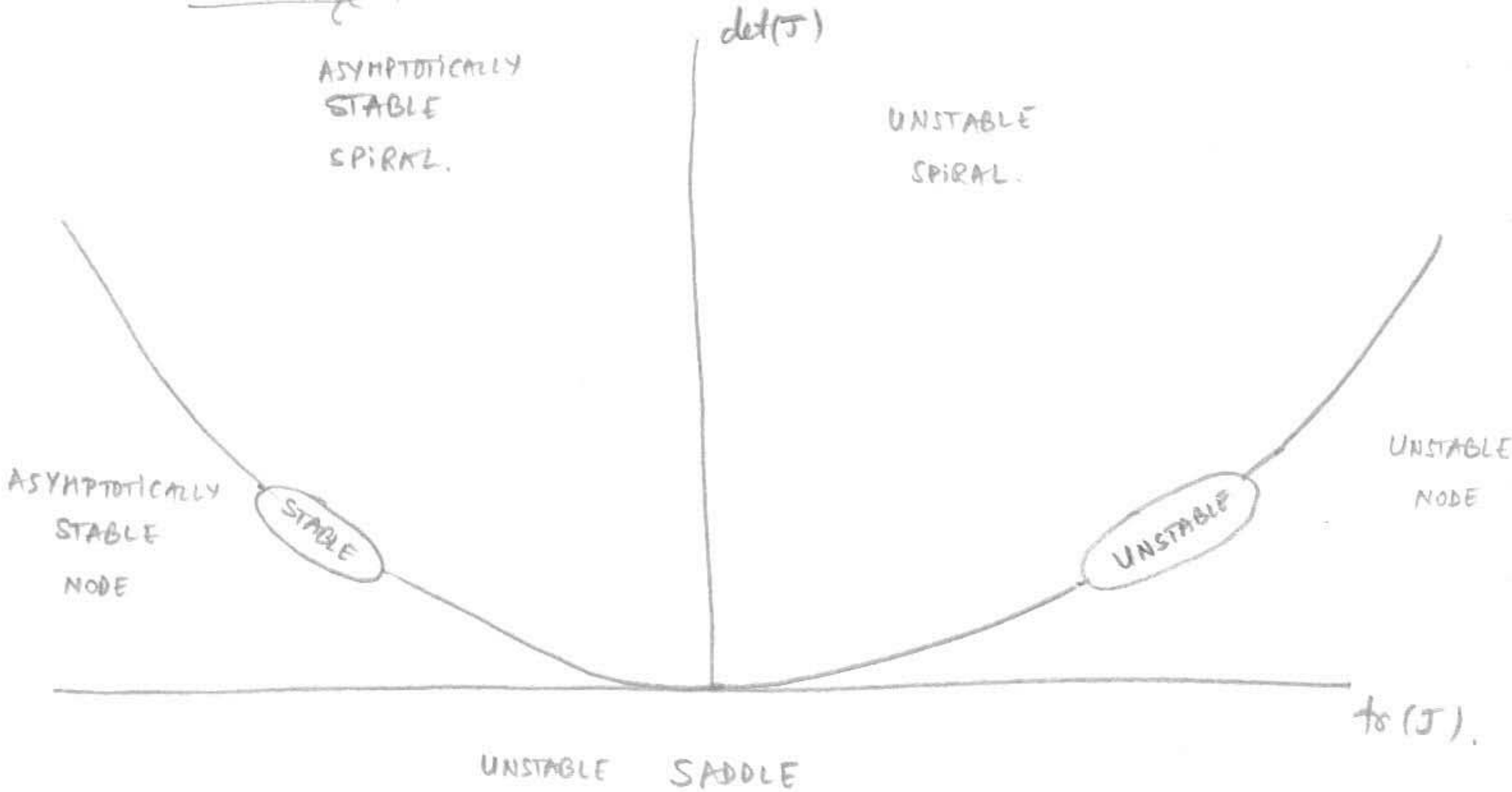


Furthermore for $(tr(J), det(J))$ outside the curves:



the stability type of the origin for LLS coincides with the stability type of $\underline{x}^0$ for $(*)$

To Summarize:



Example Consider:

$$\begin{cases} \dfrac{dx}{dt} = 4-2y \\ \dfrac{dy}{dt} = 12-3x^2 \end{cases}$$

Critical points:

$$\begin{cases} 4-2y=0 \\ 12-3x^2=0 \end{cases} \longrightarrow \begin{cases} y=2 \\ x^2=4 \end{cases} \rightarrow x=\pm 2 \qquad A_1(2,2),\ A_2(-2,2)$$

Stability analysis:

$A_1\ (2,2)$: $\quad J_1 = \begin{bmatrix} 0 & -2 \\ -6\cdot 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix}$

$$s^2 - 0\cdot s - 24 = 0 \Rightarrow s_1 = \sqrt{24},\ s_2 = -\sqrt{24}$$

$\Rightarrow A_1$ is unstable saddle

$A_2\ (-2,2)$ $\quad J_2 = \begin{bmatrix} 0 & -2 \\ 12 & 0 \end{bmatrix}$: $\quad s^2+24=0 \Rightarrow s_{1,2} = \pm i\sqrt{24}$.

A_2: we cannot conclude its stability

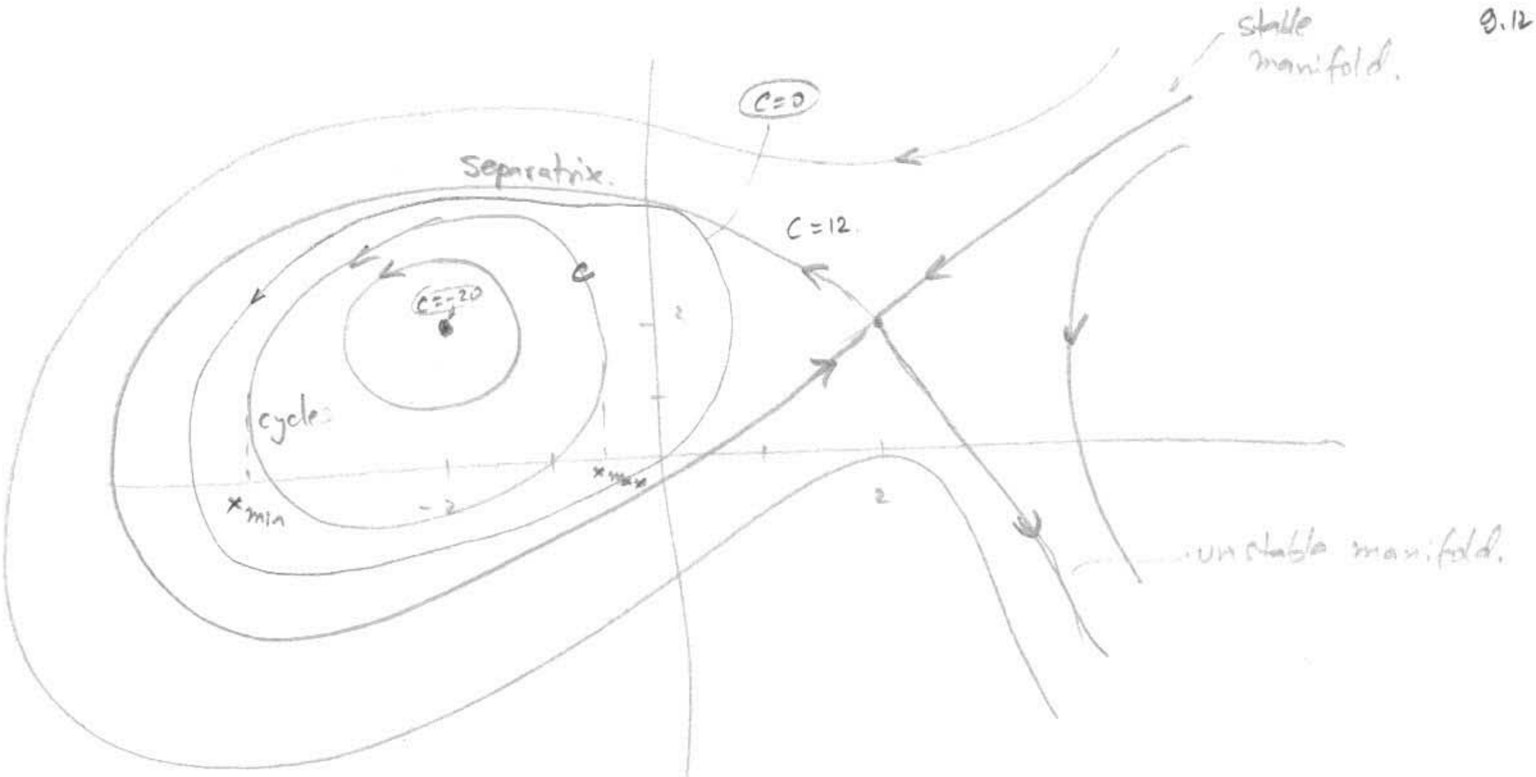
Find trajectories:

$$\frac{dx}{4-2y} = \frac{dy}{12-3x^2} \qquad \text{(separable 1}^{st}\text{ order equation)}$$

$$(12-3x^2)\,dx = (4-2y)\,dy$$

$$12x - x^3 = 4y - y^2 + C$$

$$H(x,y) = y^2 - 4y + 12x - x^3 = C = \text{const. along trajectories}$$



Here the separatrix is also a homoclinic orbit

cycles: periodic orbits.

Separatrix: $H(x,y) = H(2,2) = 4-8+24-8 = 12.$

$$y^2-4y+12x-x^3=12.$$

$H(-2,2) = 4+8-24+8 = -20.$

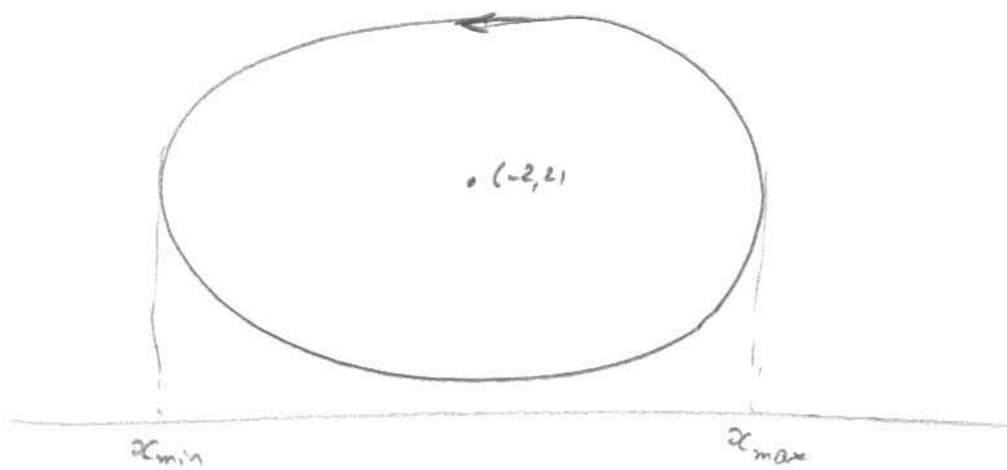
$y^2-4y+12x-x^3=-20 \to$ defines a unique point $(-2,2)$

For $-20 < C < 12$, $H(x,y)=C$ defines a closed and bounded curve
$\Rightarrow$ cycles (periodic orbits).

$$y^2-4y+12x-x^3-C=0.$$

$$\Delta = 16-4(12x-x^3-C) = 4x^3-48x+16+4C.$$

For a given C :



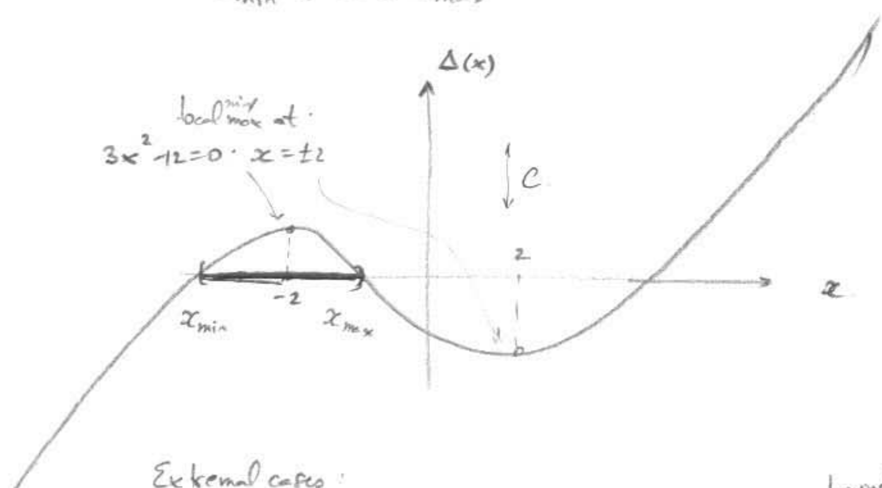
Range of x :

$$4x^3 - 48x + 16 + 4C \geq 0$$

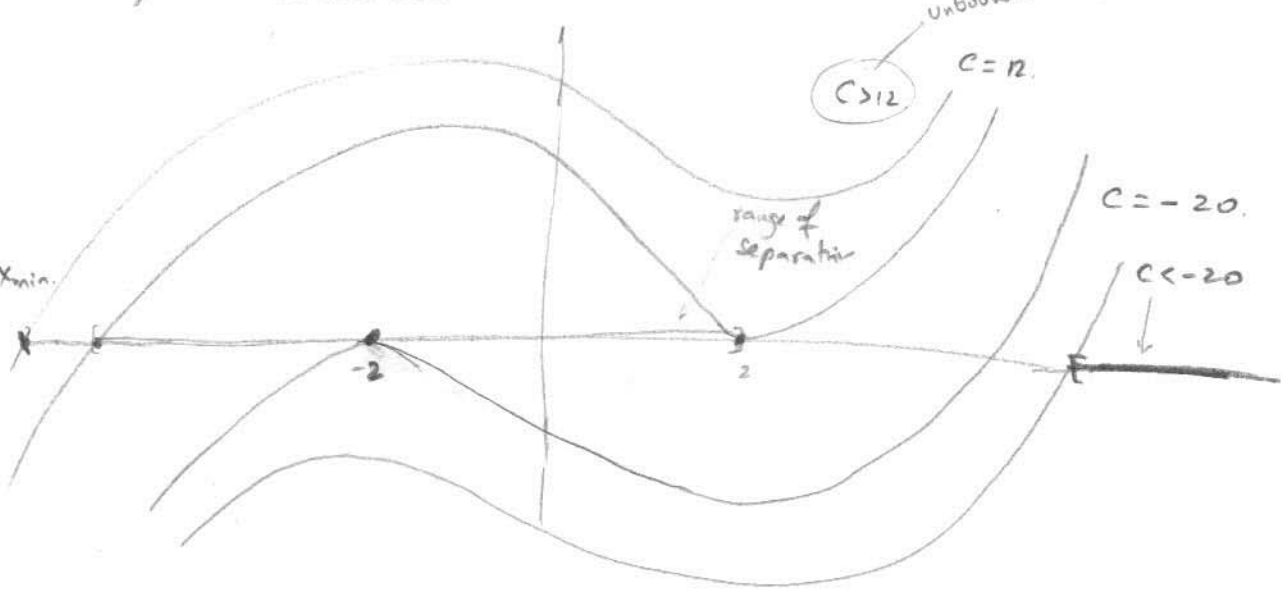
x_{min}, x_{max} solutions of $4x^3 - 48x + 16 + 4C = 0$

$$x^3 - 12x + 4 + C = 0$$

$x_{min} < -2 < x_{max}$

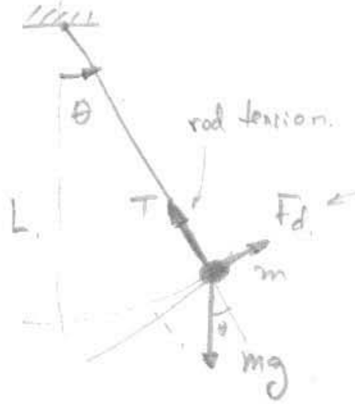


Extremal cases :



Example: PENDULUM

massless rod of length L, that can rotate about one end.



$$F_d = - C \cdot \frac{d\theta}{dt}$$

Dynamics: $F_t = - mg \sin\theta + F_d = m \frac{d^2(L\theta)}{dt^2}$

$$mL \frac{d^2\theta}{dt^2} - F_d + mg \sin(\theta) = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{C}{mL} \frac{d\theta}{dt} + \frac{g}{L} \sin(\theta) = 0 .$$

Let $\gamma = \frac{C}{mL}$, $\omega^2 = \frac{g}{L}$

$$\boxed{\frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega^2 \sin(\theta) = 0}$$

$\Rightarrow$ $x_1 = \theta$

$x_2 = \frac{d\theta}{dt}$

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = - \omega^2 \sin(x_1) - \gamma x_2 \end{cases}$$

The Oscillating Pendulum : NO dissipation

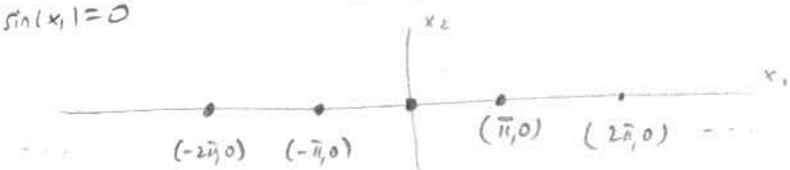
If $\gamma = 0$:
$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -\omega^2 \sin(x_1) \end{cases}$$

Equilibrium points:

$x_2 = 0$

$-\omega^2 \sin(x_1) = 0$ $\Rightarrow$ $\begin{cases} x_1 = k\pi \\ x_2 = 0 \end{cases}$ $(k\pi, 0)$ $k \in \mathbb{Z}$ (integers)

$k = \ldots, -1, 0, 1, \ldots$



Linearizations:

$$J(x) = \begin{bmatrix} 0 & 1 \\ -\omega^2\cos(x_1) & 0 \end{bmatrix} \quad ; \quad J\big((k\pi, 0)\big) = \begin{bmatrix} 0 & 1 \\ -(-1)^k\omega^2 & 0 \end{bmatrix}$$

$$tr(J) = 0$$
$$det(J) = (-1)^k\omega^2$$

$k = 2p$ (even): $\begin{matrix} tr(J) = 0 \\ det(J) = \omega^2 \end{matrix}$ $\Rightarrow$ $\lambda_{1,2} = \pm i\omega$ $\Rightarrow$ center for LLS

$k = 2p+1$ (odd): $\begin{matrix} tr(J) = 0 \\ det(J) = -\omega^2 \end{matrix}$ $\Rightarrow$ $\lambda_1 = \omega$, $\lambda_2 = -\omega$. saddle.

Trajectories:

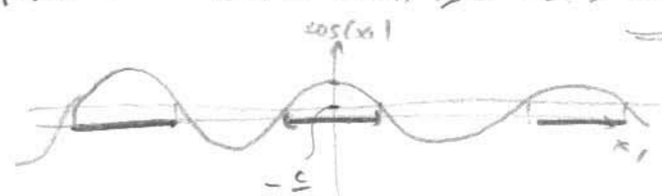
$$\frac{dx_1}{x_2} = dt = -\frac{dx_2}{\omega^2 \sin(x_1)}$$

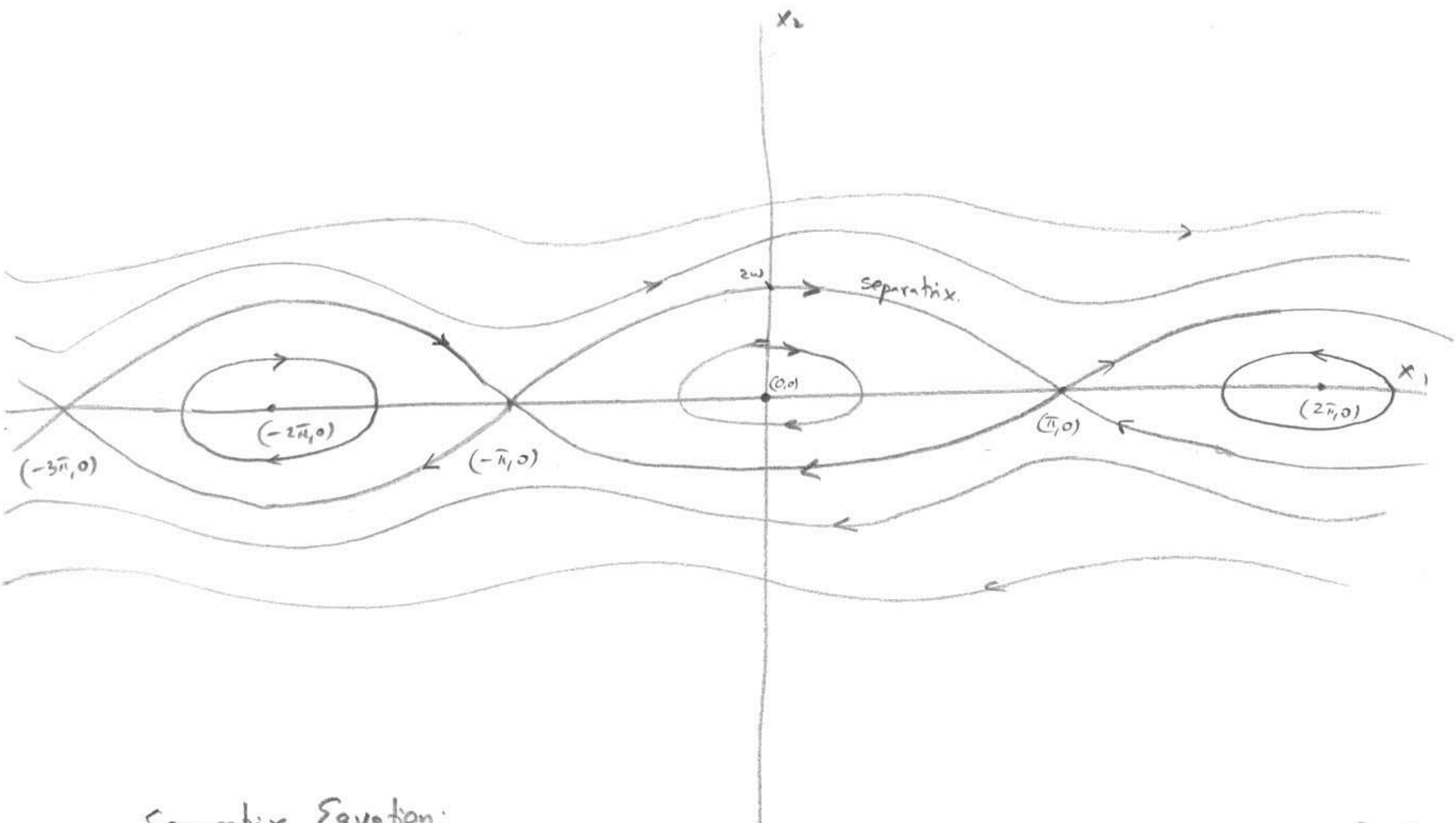
$\omega^2 \sin(x_1)\, dx_1 + x_2\, dx_2 = 0$: separable

$$H(x_1, x_2) = -\omega^2\cos(x_1) + \frac{x_2^2}{2} = C.$$

$$x_2^2 = 2\left(C + \omega^2\cos(x_1)\right) \qquad x_2 = \pm\sqrt{2(C + \omega^2\cos(x_1))}$$

For $|c| < \omega^2$: $C + \omega^2\cos(x_1) \geq 0$: $1 \geq \cos(x_1) \geq -\dfrac{C}{\omega^2}$





Separatrix Equation:

$$H(x_1, x_2) = H(\pi, 0) = \omega^2$$

$$x_2 = \pm\sqrt{2\omega^2 \underbrace{(1+\cos(x_1))}_{2\cos^2(\frac{x_1}{2})}} = \pm 2\omega \cdot \cos\left(\frac{x_1}{2}\right)$$

For $\mathbb{R}^2$ phase space,
separatrix is a heteroclinic

If $\underbrace{T^1 \times \mathbb{R}}_{\text{cylinder}}$ is the phase space
separatrix becomes a homoclinic.

Damped Pendulum

Consider the case $\gamma > 0$ in

$$\frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega^2 \sin(\theta) = 0 .$$

or: $x_1 = \theta$, $x_2 = \frac{d\theta}{dt}$

$$\begin{cases} x_1' = x_2 \\ x_2' = -\omega^2 \sin(x_1) - \gamma x_2 \end{cases}$$

Assume γ is <u>small</u>.

Equilibrium points.

1. $\begin{cases} x_2 = 0 \\ -\omega^2 \sin(x_1) - \gamma x_2 = 0 \end{cases} \longrightarrow \begin{matrix} x_2 = 0 \\ x_1 = k\pi \end{matrix}$ $\{(k\pi, 0), \ k \in \mathbb{Z}\}$.

(same as before).

2. Linearization:

$$J(k\pi, 0) = \begin{bmatrix} 0 & 1 \\ -\omega^2\cos(k\pi) & -\gamma \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2(-1)^k & -\gamma \end{bmatrix}$$

Eigenvalues/vectors:

$$\begin{cases} tr(J) = -\gamma \\ det(J) = \omega^2(-1)^k \end{cases}$$

$$\lambda^2 + \gamma\lambda + (-1)^k\omega^2 = 0$$

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4(-1)^k\omega^2}}{2}$$

Assume: $\gamma^2 < 4\omega^2 \iff \boxed{\gamma < 2\omega}$

Thus: $k = 2p$: $\lambda_{1,2} = -\frac{\gamma}{2} \pm i\sqrt{\omega^2 - \frac{\gamma^2}{4}}$ $\Rightarrow$ asympt. stable clockwise spirals.

$k = 2p+1$: $\lambda_{1,2} = -\frac{\gamma \pm \sqrt{\gamma^2 + 4\omega^2}}{2}$ $\Rightarrow$ real, opposite signs (SADDLE)

Eigenvectors

$k = 2p+1$: $\quad \lambda_1 = -\dfrac{\gamma + \sqrt{\gamma^2 + 4\omega^2}}{2} < 0, \quad \lambda_2 = \dfrac{\sqrt{\gamma^2+4\omega^2} - \gamma}{2} > 0$

$$J \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \lambda_{1,2} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad u_2 = \lambda_{1,2} u_1$$

$$\Rightarrow \lambda_1 \longrightarrow \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}_{<0}, \quad \lambda_2 \longrightarrow \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}_{>0}$$

For LLS:

Phase portrait: