# Chapter 9: Nonlinear Differential Equations and Stability

## §9.1: The Phase Plane for Linear Systems

$$\frac{d\underline{x}}{dt} = A\underline{x} \qquad , \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Equilibrium points:

Want constant solutions: $\underline{x}(t) = \underline{x}_0$, all $t$.

$$\frac{d\underline{x}}{dt} = 0 = A\,\underline{x}(t) = A\,\underline{x}_0$$

Thus $\underline{x}_0 \in N(A) = \{ \underline{z} \in \mathbb{R}^2 \text{ such that } A\underline{z} = 0 \}$ (kernel, or null space of $A$)

$$P_A(s) = \det(A - sI) = \begin{vmatrix} a-s & b \\ c & d-s \end{vmatrix} = s^2 - (a+d)s + ad - bc = s^2 - tr(A)\cdot s + \det(A)$$

$tr(A) = a+d$, $\det(A) = ad - bc$.

Case 1: $\det A = ad - bc \neq 0$.

Then the only constant solution is $\underline{x}(t) = 0$.

Case 2: $\det A = 0$. : $\boxed{\text{Non isolated critical points}}$

Then there are $\infty$-many distinct constant solutions.

$$A\underline{v} = 0.$$

admits a nonzero solution $\underline{v}$. What about a second solution?

$\boxed{2.1}$ If there are two independent $\underline{v}^1, \underline{v}^2$ such that $A\underline{v}^1 = A\underline{v}^2 = 0$

then $A = 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Phase space:

and $\underline{x}(t) = \underline{x}(0)$, all $t$, all $\underline{x}(0)$

Every point is an equilibrium point.

$tr(A) = 0$
$\det(A) = 0$

$$N(A) = \mathbb{R}^2.$$

$\boxed{2.2}$ If there is only one independant eigenvector associated to $0$, for $A$,

$\underline{v}: \; A\underline{v} = 0$.

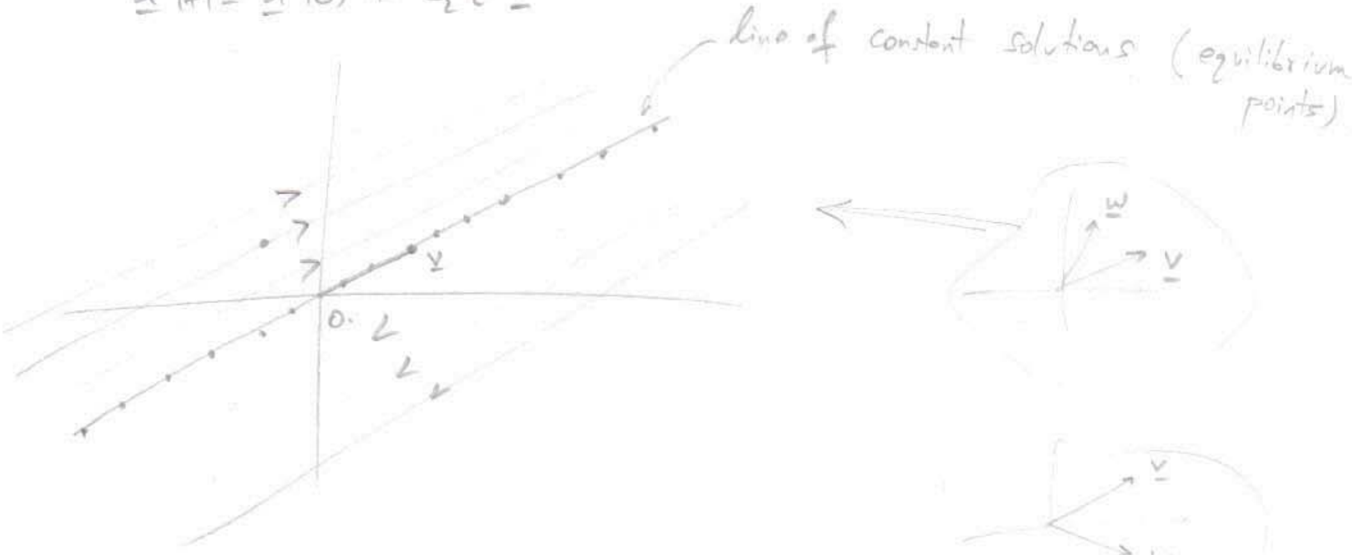
$\boxed{2.2.1}$ If $\operatorname{tr}(A) = 0$, $p_A(s) = s^2 \rightarrow 0$ is a repeated eigenvalue.

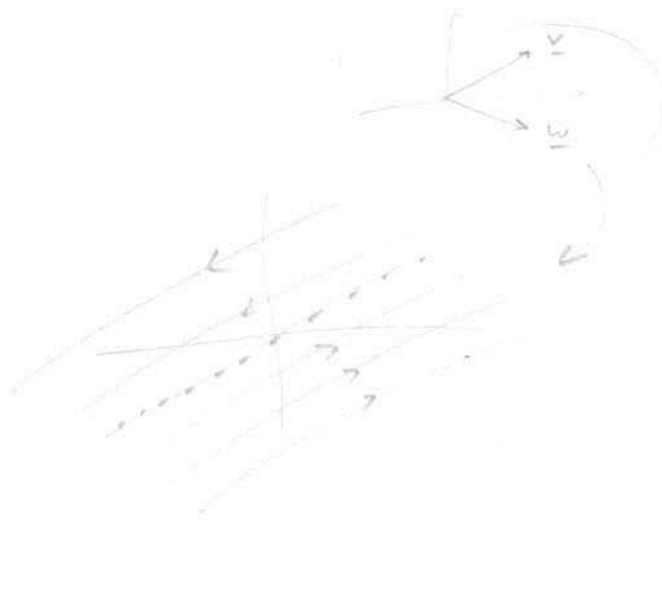
Use generalized eigenvector theory: $\underline{x}^{(2)}(t) = t\,\underline{v} + \underline{w}$.

$$\left.\begin{array}{l} \dfrac{d}{dt}\underline{x} = \underline{v} \\ A\underline{x} = t\underbrace{A\underline{v}}_{0} + A\underline{w} \end{array}\right\} \quad A\underline{w} = \underline{v} \;\rightarrow \text{solve for } \underline{w}.$$

General solution. $\underline{x}(t) = c_1\underline{v} + c_2(t\underline{v} + \underline{w}) = \underbrace{c_1\underline{v} + c_2\underline{w}}_{\underline{x}(0)} + c_2 t\underline{v}$

$\underline{x}(t) = \underline{x}(0) + c_2 t\underline{v}$



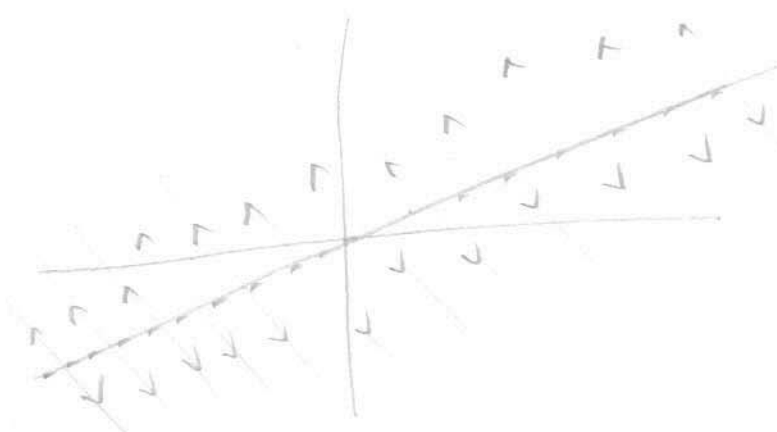



$\boxed{2.2.2.}$ $tr(A) \neq 0$: $\quad p_A(s) = s^2 - tr(A)\,s = s\,[s - tr(A)]$

$$\Rightarrow \quad s_1 = 0, \quad s_2 = tr(A) \neq 0$$

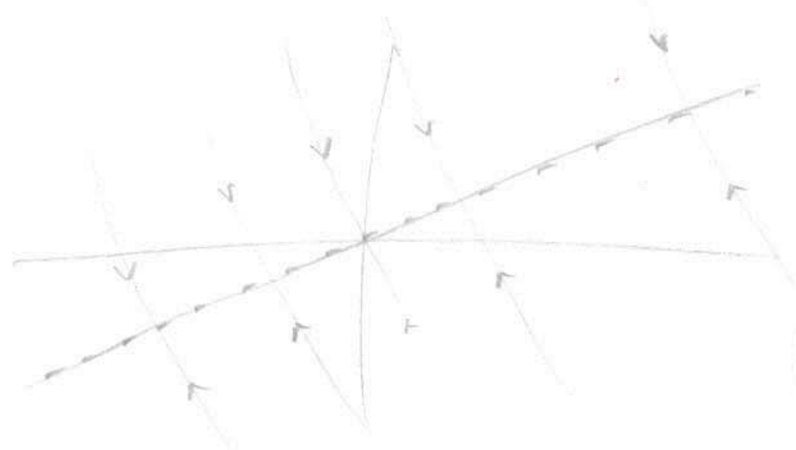
We have a second eigenvector: $A\,\underline{v}^2 = s_2 \cdot \underline{v}^2$

General solution: $\quad \underline{x}(t) = c_1\,\underline{v} + c_2\,\underline{v}^2\,e^{tr(A)\cdot t}$



$tr(A) > 0$

Origin is unstable



$tr(A) < 0$

Origin is stable but not asymptotically stable

Case 1 : $\det(A) = ad - bc \neq 0$

$$p_A(s) = s^2 - \operatorname{tr}(A)\cdot s + \det(A) = 0 \implies s_1, s_2 \text{ and } s_1 \neq 0,\ s_2 \neq 0.$$

$$s_{1,2} = \frac{\operatorname{tr}(A) \pm \sqrt{(\operatorname{tr}(A))^2 - 4\det(A)}}{2}$$

$$\boxed{\begin{aligned} s_1 + s_2 &= \operatorname{tr}(A) \\ s_1 \cdot s_2 &= \det(A) \end{aligned}}$$

distinct eigenvalues : $(\operatorname{tr}(A))^2 - 4\det(A) \neq 0$.

repeated eigenvalues : $(\operatorname{tr}(A))^2 - 4\det(A) = 0$.

$\boxed{1.1}$ Distinct Eigenvalues : $s_1 \neq s_2$.

$\boxed{1.1.1}$ Real & distinct eigenvalues : $\Delta = (\operatorname{tr}(A))^2 - 4\det(A) > 0$.

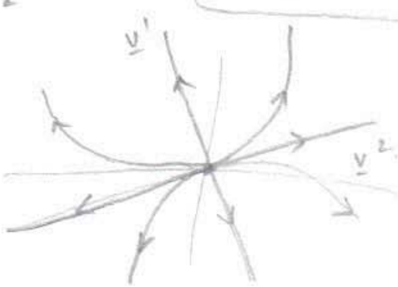
$$\det(A) < \tfrac{1}{4}(\operatorname{tr}A)^2.$$

$$\Rightarrow \underline{x}(t) = c_1 e^{s_1 t}\underline{v}^1 + c_2 e^{s_2 t}\underline{v}^2 \quad : \text{where} \quad \begin{cases} A\underline{v}^1 = s_1\underline{v}^1 \\ A\underline{v}^2 = s_2\underline{v}^2 \end{cases}$$

a). $s_1, s_2 > 0$ : $\boxed{\text{Unstable NODE}}$

Say $s_1 > s_2 > 0$

faster mode ($s_1$), slow mode ($s_2$).

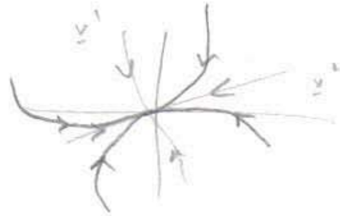



$\boxed{\operatorname{tr}(A) > 0,\ \det(A) > 0}$

b) $s_1, s_2 < 0$

say $s_1 < s_2 < 0$.

$\boxed{\text{Asymptotically stable NODE}}$




$\boxed{\operatorname{tr}(A) < 0,\ \det(A) > 0}$

c) $s_1 < 0 < s_2$ : $\boxed{\text{SADDLE}}$

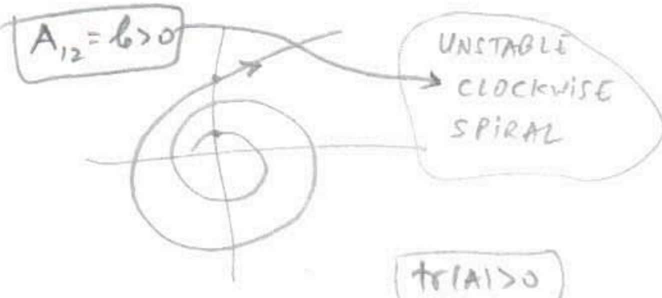

$\boxed{\det(A) < 0}$ ($\operatorname{tr}(A)$ could be $> = < 0$)

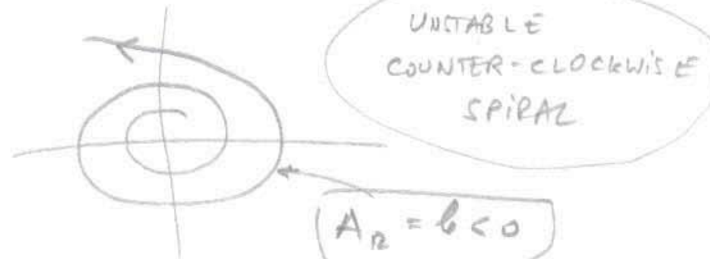
# $\boxed{1.1.2}$ Complex eigenvalues.

$$s_{1,2} = \lambda \pm i\mu. \quad \Rightarrow \quad \underline{x}(t) = c_1 e^{\lambda t}\, Re\left[e^{i\mu}\underline{v}'\right] + c_2 e^{\lambda t}\, Im\left[e^{i\mu}\underline{v}'\right].$$

$2\lambda = tr(A)$, $\lambda^2 + \mu^2 = det(A)$

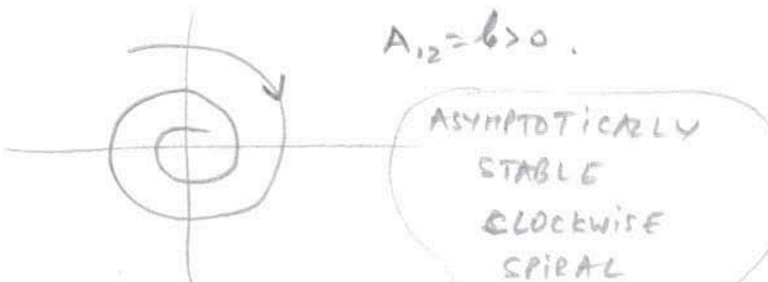
a) $\lambda > 0$ : $\boxed{\text{UNSTABLE SPIRAL}}$



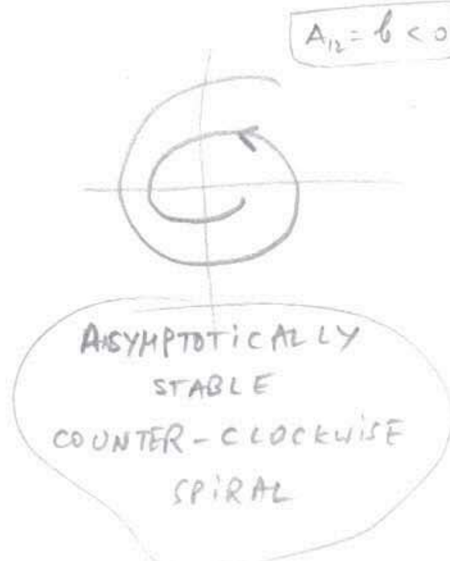




$\boxed{tr(A) > 0}$

b) $\lambda < 0$ : $\boxed{\text{ASYMPTOTICALLY STABLE SPIRAL}}$




or




$\boxed{2\lambda = tr(A) < 0}$

c) $\lambda = 0$ : $tr(A) = 0$, $det(A) = \mu^2 > 0$

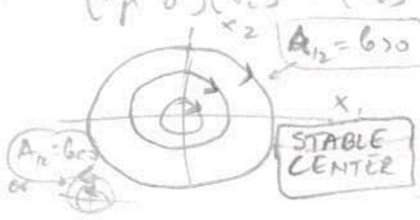
Then: $\underline{x}(t) = c_1\, Re\left[e^{i\mu t}\underline{v}\right] + c_2\, Im\left[e^{i\mu t}\underline{v}\right]$

Typically: $A = \begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix} \rightsquigarrow$ eigenvector: $\begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = i\mu\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $\quad \mu v_2 = i\mu v_1$; $v_2 = iv_1$

$\underline{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\underline{x}^1(t) = Re\left[e^{i\mu t}\begin{bmatrix} 1 \\ i \end{bmatrix}\right] = \begin{bmatrix} \cos(\mu t) \\ -\sin(\mu t) \end{bmatrix}$

$\underline{x}^2(t) = Im\left[e^{i\mu t}\begin{bmatrix} 1 \\ i \end{bmatrix}\right] = \begin{bmatrix} \sin(\mu t) \\ \cos(\mu t) \end{bmatrix}$




$\underline{x}(t) = \begin{bmatrix} c_1\cos(\mu t) + c_2\sin(\mu t) \\ -c_1\sin(\mu t) + c_2\cos(\mu t) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$\alpha^2 + \beta^2 = c_1^2 + c_2^2$

$\boxed{1.2}$ Repeated Eigenvalues $\quad s_1 = s_2 \neq 0$.

$$tr(A) = 2s_1.$$

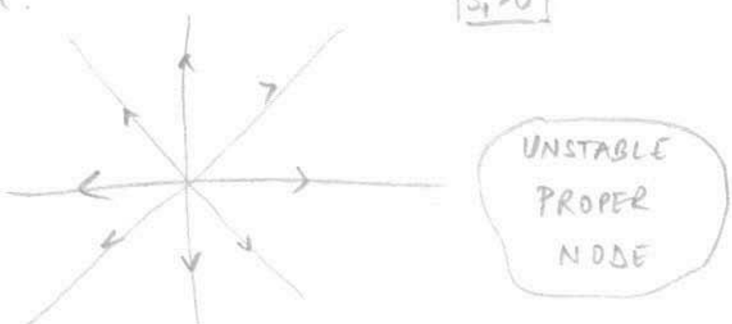
**1.2.1:** If there are two independent eigenvectors $\underline{v}^1, \underline{v}^2$

then $\underline{x}(t) = c_1 e^{s_1 t} \underline{v}^1 + c_2 e^{s_1 t} \underline{v}^2 = e^{s_1 t} \underbrace{\left[ c_1 \underline{v}^1 + c_2 \underline{v}^2 \right]}_{\underline{x}(0)} = e^{s_1 t} \underline{x}(0)$

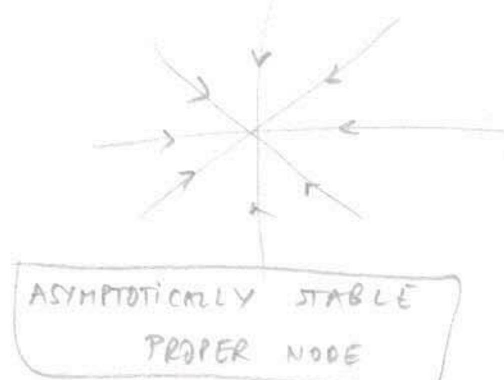
Thus $A = s_1 I$ (must be!)

$$= \begin{pmatrix} s_1 & 0 \\ 0 & s_1 \end{pmatrix}.$$

and. $\boxed{s_1 > 0}$ or $\boxed{s_1 < 0}$



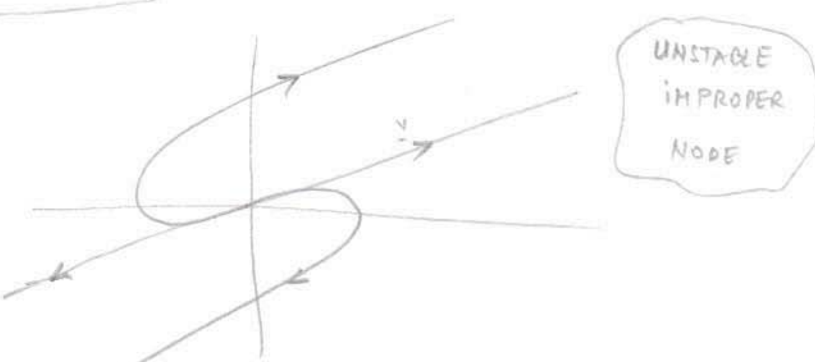

$tr(A) > 0$.

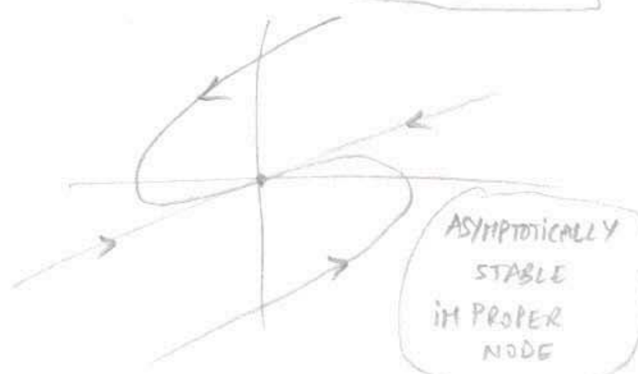


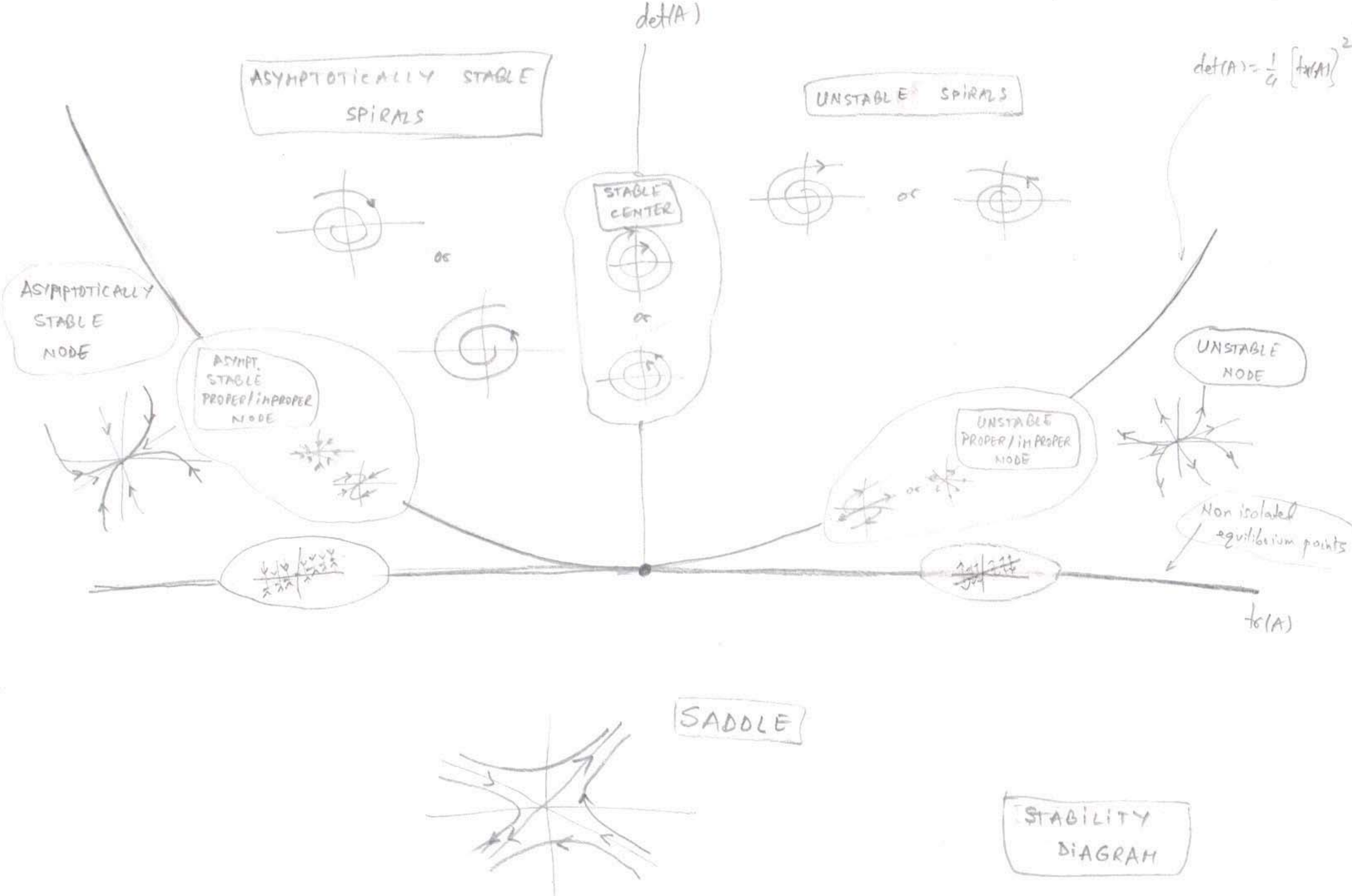

$tr(A) < 0$.

**1.2.2:** If there only one independent eigenvector $\underline{v}$, then the second solution is $\underline{x}^2(t) = t e^{s_1 t} \cdot \underline{v} + e^{s_1 t} \underline{w}$

$\boxed{s_1 > 0 \cdot tr(A) > 0}$ $\qquad$ $\boxed{s_1 < 0, \ tr(A) < 0}$




det(A)
ASYMPTOTICALLY STABLE SPIRALS
UNSTABLE SPIRALS
$\det(A) = \frac{1}{4}[tr(A)]^2$
or
STABLE CENTER
ASYMPTOTICALLY STABLE NODE
ASYMPT. STABLE PROPER/IMPROPER NODE
UNSTABLE NODE
UNSTABLE PROPER/IMPROPER NODE
Non isolated equilibrium points
tr(A)
SADDLE
STABILITY DIAGRAM

9.7