Solutions of the third In-Class Exam

MATH 246, Professor Radu Balan

1. [10pts] Compute the inverse Laplace transform of $F(s)=\frac{1}{s^3-1}$.

Solution

Note

$$F(s)=\frac{1}{(s-1)(s^2+s+1)}=\frac{1}{3}\frac{1}{s-1}-\frac{1}{3}\frac{(s+2)}{s^2+s+1}=\frac{1}{3}\frac{1}{s-1}-\frac{1}{3}\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}-\frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}$$

Thus $f(t)=\frac{1}{3}e^t u(t)-\frac{1}{3}e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$

2. [9pts] Transform the equation $(1+t^2)\frac{d^3u}{dt^3}+\frac{du}{dt}-u^2=\sin(t)$ into a system of first order (possibly nonlinear) differential equations.

Solution

Set $x_1=u, x_2=\frac{du}{dt}, x_3=\frac{d^2u}{dt^2}$.

Then the equation turns into

$$\frac{dx_1}{dt}=x_2$$
$$\frac{dx_2}{dt}=x_3$$
$$\frac{dx_3}{dt}=\frac{x_1^2}{1+t^2}-\frac{x_2}{1+t^2}+\frac{\sin(t)}{1+t^2}$$

3. [10pts] Find the Laplace transform Y(s) of the solution y(t) of the initial-value problem

$$\frac{d^3y}{dt^3}+3\frac{dy}{dt}-2y=f(t)\ ,\quad y(0)=1,\ y'(0)=2,\ y''(0)=-1$$

where

$$f(t)=\begin{cases}\cos(t) & 0\le t<2\\ t & 2\le t\end{cases}.$$

You may use the table on the last page. DO NOT take the inverse Laplace transform to find y(t), just solve for Y(s) !

Solution

$$s^3Y(s) - s^2 - 2s + 1 + 3sY(s) - 3 - 2Y(s) = F(s)$$

Thus $Y(s) = \dfrac{F(s) + s^2 + 2s + 2}{s^3 + 3s - 2}$

Now:

$$f(t) = \cos(t)u(t) + (t - \cos(t))u(t-2) = \cos(t)u(t) + (t - 2 + 2 - \cos(t - 2 + 2))u(t-2) =$$
$$= \cos(t)u(t) + (t-2)u(t-2) + 2u(t-2) - \cos(2)\cos(t-2)u(t-2) + \sin(2)\sin(t-2)u(t-2)$$

Thus

$$F(s) = \frac{s}{s^2+1} + e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s} - \frac{\cos(2)s}{s^2+1} + \frac{\sin(2)}{s^2+1}\right)$$

and

$$Y(s) = \frac{1}{s^3+3s-2}\left[\frac{s}{s^2+1} + e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s} - \frac{\cos(2)s}{s^2+1} + \frac{\sin(2)}{s^2+1}\right)\right] + \frac{s^2+2s+2}{s^3+3s-2}$$

4. [15pts] Consider the vector-valued functions $x_1(t) = \begin{pmatrix} 2t+1 \\ 1 \end{pmatrix}$ and $x_2(t) = \begin{pmatrix} t^2-1 \\ t \end{pmatrix}$.

a. [5pts] Compute the Wronskian $W[x_1,x_2](t)$ and find the maximal interval I=(a,b) containing 0 so that $W[x_1,x_2](t) \neq 0$ for all a<t<b.

b. [5pts] Find A(t) such that x1(t) , x2(t) form a fundamental set of solutions to the system $\dfrac{dx}{dt} = A(t)x$

c. [5pts] For the system found at b) solve the initial-value problem

$$\frac{dx}{dt} = A(t)x \quad , \quad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solution

a. The Wronskian is

$$W[x_1,x_2](t) = \begin{vmatrix} 2t+1 & t^2-1 \\ 1 & t \end{vmatrix} = 2t^2 + t - t^2 + 1 = t^2 + t + 1$$

Note $W(t) \neq 0$ for all real t. Hence I=R.

b. Compute A(t) from

$$\begin{pmatrix} 2 & 2t \\ 0 & 1 \end{pmatrix} = A(t)\begin{pmatrix} 2t+1 & t^2-1 \\ 1 & t \end{pmatrix} \Rightarrow A(t) = \frac{1}{t^2+t+1}\begin{pmatrix} 2 & 2t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 1-t^2 \\ -1 & 2t+1 \end{pmatrix} = \frac{1}{t^2+t+1}\begin{pmatrix} 0 & 2t^2+2t+2 \\ -1 & 2t+1 \end{pmatrix}$$

c. The solution is given by:

$$x(t) = \begin{pmatrix} 2t+1 & t^2-1 \\ 1 & t \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2t+1 & t^2-1 \\ 1 & t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2t+1 & t^2-1 \\ 1 & t \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2t+1 \\ 1 \end{pmatrix} = x_1(t)$$

5. [20pts] Consider the linear system x'=Ax where $A = \begin{pmatrix} 2 & 8 \\ 2 & 2 \end{pmatrix}$.

a. [5pts] Compute e^{tA}

b. [5pts] Solve the initial value problem x'=Ax, $x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

c. [5pts] Sketch the trajectory $(x(t), y(t))$ in the phase plane for $0 \le t \le 100$ indicating the starting point and the end point.

d. [5pts] For what values of the initial condition x(0) the trajectory $\{x(t)\}$ of this linear system remains bounded? What is the limit $\lim_{t\to\infty} x(t)$ in this case?

Solution

a. The characteristic polynomial: $p_A(s) = s^2 - 4s - 12 = (s-6)(s+2)$. Thus r_1=6 and r_2=-2 are the eigenvalues. Their corresponding eigenvectors are:

$$0 = (A-6I)v = \begin{pmatrix} -4 & 8 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow -4a+8b = 0 \Rightarrow a = 2b \Rightarrow v^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$0 = (A+2I)v = \begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow 4a+8b = 0 \Rightarrow a = -2b \Rightarrow v^2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Thus a fundamental matrix is

$$\Phi(t) = \begin{pmatrix} 2e^{6t} & -2e^{-2t} \\ e^{6t} & e^{-2t} \end{pmatrix}$$

And the matrix exponential is

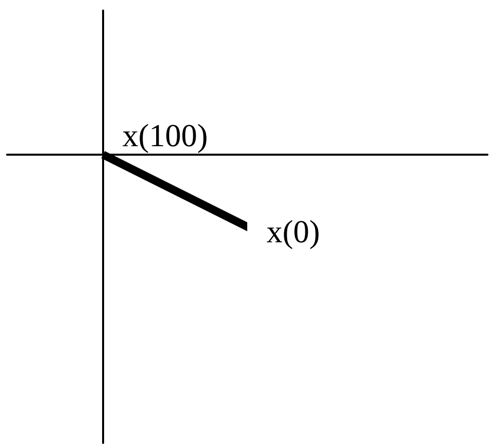
$$e^{tA} = \Phi(t)\Phi^{-1}(0) = \begin{pmatrix} 2e^{6t} & -2e^{-2t} \\ e^{6t} & e^{-2t} \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 2e^{6t} & -2e^{-2t} \\ e^{6t} & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{6t} + \frac{1}{2}e^{-2t} & e^{6t} - e^{-2t} \\ \frac{1}{4}e^{6t} - \frac{1}{4}e^{-2t} & \frac{1}{2}e^{6t} + \frac{1}{2}e^{-2t} \end{pmatrix}$$

b. The IVP has solution

$$x(t) = \begin{pmatrix} \frac{1}{2}e^{6t} + \frac{1}{2}e^{-2t} & e^{6t} - e^{-2t} \\ \frac{1}{4}e^{6t} - \frac{1}{4}e^{-2t} & \frac{1}{2}e^{6t} + \frac{1}{2}e^{-2t} \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2e^{-2t} \\ -e^{-2t} \end{pmatrix} = e^{-2t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

One can check this is the solution. If by checking it turns out this is not a solution then one can solve this IVP independently of a) and then trace back the error (either in part a) or in part b).

c. The trajectory is a segment of the line defined by the second eigenvector. It starts at (2,-1) (the initial condition) and for t=100 (e^{-200} is approximated by 0) it ends very close to the origin (0,0).



d. Since the origin is a saddle, the only bounded trajectories are those initialized on the line generated by the eigenvector associated to a negative eigenvalue. Specifically, the locus of the initial conditions is the line $\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix} a, -\infty < a < \infty\right\}$. In this case any such trajectory converges to the origin. Hence $\lim_{t\to\infty} x(t) = 0$.

6. [20pts] Solve each of the following initial-value problems:

a. [10pts] $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b. [10pts] $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -4 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Solution

a. *Laplace method:* Apply Laplace transform:

$$s\begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} \Rightarrow \begin{pmatrix} s+2 & 2 \\ 5 & s-1 \end{pmatrix}\begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus:

$$\begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} s+2 & 2 \\ 5 & s-1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{s^2+s-12}\begin{pmatrix} s-1 & -2 \\ -5 & s+2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{s-3}{(s-3)(s+4)} \\ \frac{s-3}{(s-3)(s+4)} \end{pmatrix} = \frac{1}{s+4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

And therefore: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{=4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eigenvectors based method: Compute the characteristic polynomial: p(s)=s^2+s-12=(s-3)(s+4) and hence the two eigenvalues 3 and -4. The corresponding eigenvectors are $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Hence the general solution is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{3t}\begin{pmatrix} -2 \\ 5 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Using the initial condition we obtain c_1=0, c_2=1, which yields the same solution as above, $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b.

$$p_A(s) = \det(A - sI) = \begin{vmatrix} -1-s & 4 \\ -4 & 7-s \end{vmatrix} = (s+1)(s-7)+16 = s^2 - 6s + 9 = (s-3)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 3$$

First eigenvector: $\begin{pmatrix} -4 & 4 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = b \Rightarrow v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The generalized eigenvector: $\begin{pmatrix} -4 & 4 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow -4w_1 + 4w_2 = 1 \Rightarrow w = -\begin{pmatrix} 1/4 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Hence two independent solutions are: $\begin{pmatrix}1\\1\end{pmatrix}e^{3t}$ and $\begin{pmatrix}1\\1\end{pmatrix}te^{3t} - \begin{pmatrix}0.25\\0\end{pmatrix}e^{3t}$

The general solution is $\begin{pmatrix}x(t)\\y(t)\end{pmatrix} = c_1 e^{3t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{3t}\begin{pmatrix}t-0.25\\t\end{pmatrix}$ and the initial condition implies

$$\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}c_1 - 0.25c_2\\c_1\end{pmatrix} \Rightarrow c_1 = 2, c_2 = -4$$

and thus the solution becomes

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix} = e^{3t}\begin{pmatrix}3-4t\\2-4t\end{pmatrix}.$$

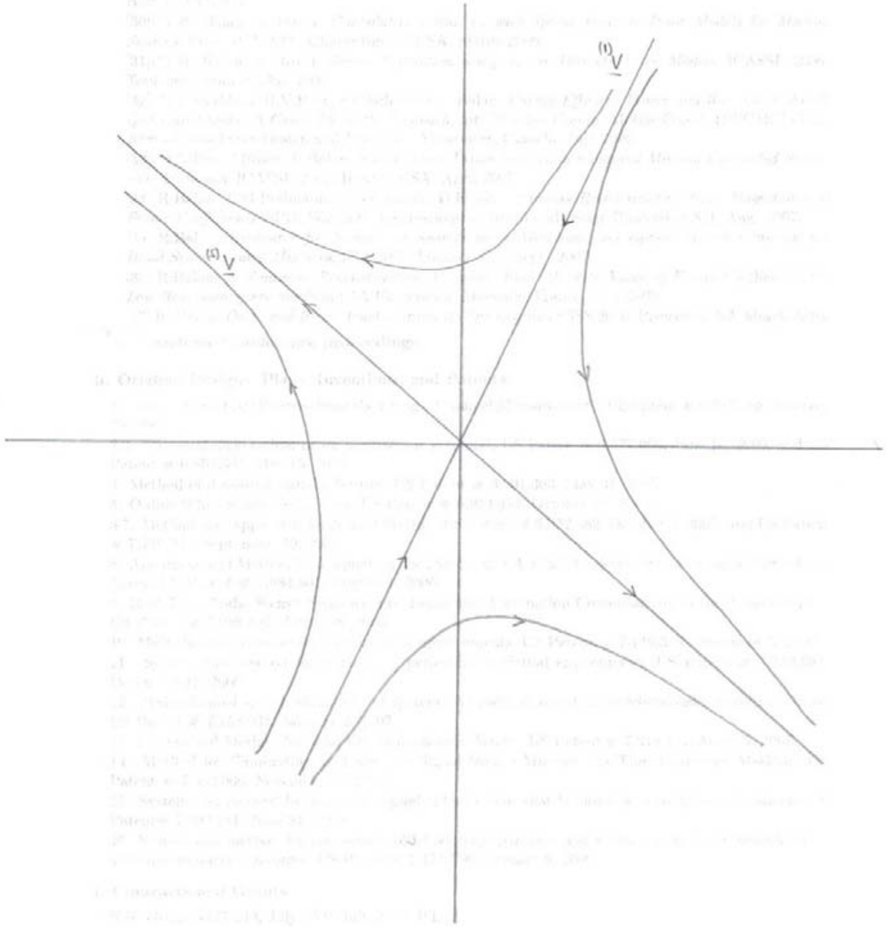
7. [16pts] Sketch the phase-plane portrait for each of the following two systems. Indicate typical trajectories. Be careful to mark any eigenvector. For each portrait identify its type and give a reason why the origin is either asymptotically stable, stable, or unstable.

a. [8pts] $\frac{d}{dt}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-2 & -2\\-5 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

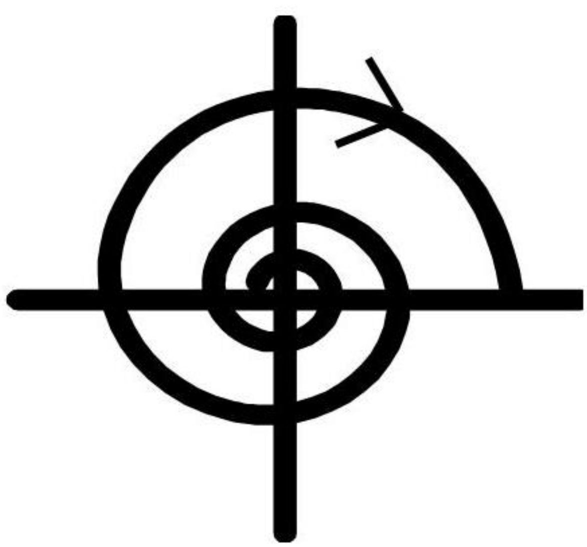
b. [8pts] $\frac{d}{dt}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}5 & 4\\-5 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

<u>Solution</u>

a. For $\frac{d}{dt}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-2 & -2\\-5 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$ the eigenvalues are r_2=3 and r_1=-4 hence the origin is a saddle. The two eigenvectors are: $v^2 = \begin{pmatrix}-2\\5\end{pmatrix}$ (unstable) and $v^1 = \begin{pmatrix}1\\1\end{pmatrix}$ (stable). Hence the phase portrait looks like



b. For $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, the origin is an unstable spiral (or spiral source), because the eigenvalues are conjugate complex with strictly positive real part. The phase portrait looks like this:



Note the trajectories are oriented ***clockwise*!** Because at (0,1), the tangent vector is $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.