(1) [4pts] A certain college graduate borrows $8000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming that interest is compounded continuously and that the borrower makes payments continuously at a constant annual rate $k$, determine the payment rate $k$ that is required to pay off the loan in 3 years. Also determine how much interest is paid during the 3-year period.

The following numbers might be useful: $\exp(0.3) = 1.3$, $\exp(0.3)/\left(\exp(0.3) - 1\right) = 3.86$. To get full credit please make all the computations.

Solution.

Let $S(t)$ denote the loan amount after $t$ years. The equation governing $S(t)$ is:

$$\frac{dS}{dt} = 0.1 \cdot S - k$$

with $S(0) = 8000$ and $k$ expressed in $$/year.

The solution to this equation is:

$$S(t) = S(0)e^{rt} - \frac{k}{r}\left(e^{rt} - 1\right) = 8000e^{0.1t} - 10k\left(e^{0.3t} - 1\right)$$

We know that $S(3) = 0$. Hence:

$$8000e^{0.3} - 10k\left(e^{0.3} - 1\right) = 0$$

$$k = \frac{800e^{0.3}}{e^{0.3} - 1} = \frac{800 \cdot 3.86}{3.86 - 1} = \$800 \cdot 3.86 / \text{year} = \$8 \cdot 3.86 / \text{year} = \$3088 / \text{year}$$

The total interest paid the 3-year period is $3k - S_0$, that is:

$Interest = $3088 \cdot 3 - $8000 = $9264 - $8000 = $1264$
(2) [6 pts] For the following differential equations: (i) determine the equilibrium points; (ii) classify each equilibrium point; and (iii) sketch the phase line portrait:

(a) \( \frac{dy}{dt} = y(y - 1)^2(y + 2) \), \( -\infty < y < \infty \)

(b) \( \frac{dy}{dt} = e^{-y} - e \), \( -\infty < y < \infty, e = 2.71 \)

(You do not have to solve these equations.)

Solutions.

Problem (a)

(i) The equilibrium points are: 0, 1, -2.

(ii) Signature of \( f(y) \):

<table>
<thead>
<tr>
<th>( y )</th>
<th>-2</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F(y) )</td>
<td>+</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

Hence: -2 is (asymptotically) stable, 0 is unstable; 1 is semistable.

(iii) The phase line portrait

Problem (b)

(i) The only equilibrium point is at: \( y = -1 \).

(ii) We need the signature of \( f(y) \) around this value:

<table>
<thead>
<tr>
<th>( Y )</th>
<th>-1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(y) )</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

Hence -1 is (asymptotically) stable.

(iii) The phase line portrait: