Quiz 6, MATH 246, Professor Radu Balan, Tuesday, 19 October 2010

1. [5pts] Find the general solution of:

$$y''+\omega_0^2 y = \cos(\omega t) \qquad , \qquad \omega_0^2 \neq \omega^2$$

<u>Solution</u>

1. The homogeneous equation is:

$y''+\omega_0^2 y = 0$

The characteristic equation is: $r^2 + \omega_0^2 = 0$ and roots $r_{1,2} = \pm i\omega_0$

Hence: $y_1(t) = \cos(\omega_0 t), y_2(t) = \sin(\omega_0 t)$

2. We use the method of undetermined coefficients to find a particular solution. We look for a solution of the form:

$Y(t) = A\cos(\omega t) + B\sin(\omega t)$

Substitute into the differential equation:

$Y'(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$

$Y''(t) = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$

$Y''+\omega_0^2 Y = A(\omega_0^2 - \omega^2)\cos(\omega t) + B(\omega_0^2 - \omega^2)\sin(\omega t) = \cos(\omega t)$

Thus:

$$A = \frac{1}{\omega_0^2 - \omega^2}, B = 0$$

And a particular solution is

$$Y(t) = \frac{1}{\omega_0^2 - \omega^2}\cos(\omega t)$$

3. The general solution of this equation is:

$$y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{1}{\omega_0^2 - \omega^2}\cos(\omega t)$$

2. [5pts] Find the solution of the initial value problem

$$y''+y'-2y=2t \quad , \quad y(0)=0 \quad , \quad y'(0)=1$$

Solution

1. First we find the solution of the homogeneous equation:

$y''+y'-2y=0$

The characteristic equation is $r^2+r-2=0$ with roots $r_1=1$, $r_2=-2$.

Hence $y_1(t)=e^t, y_2(t)=e^{-2t}$.

2. Find a particular solution of the nonhomegeneous equation. Using the method of undetermined coefficients, we choose a function:

$Y(t)=At+B$

Substitute into equation:

$A-2At-2B=2t$

Hence $A=-1, B=-0.5$

And a particular solution is

$Y(t)=-t-0.5$

3. The general solution is

$y(t)=c_1e^t+c_2e^{-2t}-t-0.5$

4. We impose the initial conditions:

$y(0)=c_1+c_2-0.5=0$

$y'(t)=c_1e^t-2c_2e^{-2t}-1$

$y'(0)=c_1-2c_2-1=1$

Thus

$$\begin{cases} c_1+c_2 = 0.5 \\ c_1-2c_2 = 2 \end{cases} \rightarrow \begin{cases} c_1 = 1 \\ c_2 = -0.5 \end{cases}$$

Hence the solution of the IVP is:

$y(t)=e^t-0.5e^{-2t}-t-0.5$