Quiz 8, MATH 246, Professor Radu Balan, Tuesday, 9 November 2010

1. [3pts] Use the definition of Laplace transform to compute the Laplace transform of

$$f(t) = \begin{cases} 1 & t \geq 6 \\ 0 & t < 6 \end{cases}$$

Solution

$$F(s) = \int_0^\infty e^{-ts} f(t)dt = \int_6^\infty e^{-ts} dt = -\frac{1}{s} \lim_{A\to\infty} \left[e^{-ts} \right]\Big|_6^A = -\frac{1}{s} \left[\lim_{A\to\infty} e^{-sA} - e^{-6s} \right] = \frac{e^{-6s}}{s}, s > 0$$

2. [7pts] Find the solution of the initial value problem:

$$\begin{cases} y'' + y = g(t) \\ y(0) = 0, y'(0) = 1 \end{cases} \quad g(t) = \begin{cases} t/2 & 0 \leq t < 6 \\ 3 & t \geq 6 \end{cases}$$

Short table: $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}, \mathcal{L}\{\cos(at)\} = \frac{s}{s^2 + a^2}$

Solution

Let *Y(s)* and *G(s)* denote the Laplace transforms of *y(t)* and, respectively, *g(t)*.
The Laplace transform of the equation is:

$$s^2 Y(s) - 1 + Y(s) = G(s)$$

Hence the Laplace transform of the solution is:

$$Y(s) = \frac{1 + G(s)}{s^2 + 1}$$

Since

$$g(t) = \frac{t}{2} u(t) + \left(3 - \frac{t}{2}\right) u(t-6) = \frac{t}{2} u(t) - \frac{t-6}{2} u(t-6)$$

It follows:

$$G(s) = \frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}$$

And hence:

$$Y(s) = \frac{1 - e^{-6s}}{2s^2(s^2+1)} + \frac{1}{s^2+1}$$

Expand in partial fractions:

$$\frac{1 - e^{-6s}}{2} \left(\frac{1}{s^2} - \frac{1}{s^2+1} \right) = \frac{1}{2} \left(\frac{1}{s^2} - \frac{1}{s^2+1} \right) - \frac{1}{2} \left(\frac{1}{s^2} - \frac{1}{s^2+1} \right) e^{-6s}$$

Note: $\mathcal{L}^{-1}\left\{\frac{1}{s^2}-\frac{1}{s^2+1}\right\}=t-\sin(t)$ and thus:

$$y(t)=\frac{1}{2}\left(t-\sin(t)\right)-\frac{1}{2}\left(t-6-\sin(t-6)\right)u(t-6)+\sin(t)=$$

$$=\frac{1}{2}\left(t+\sin(t)\right)-\frac{1}{2}\left(t-6-\sin(t-6)\right)u(t-6)=\begin{cases}\frac{1}{2}\left(t+\sin(t)\right) & 0\le t<6\\ 3+\frac{\sin(t)+\sin(t-6)}{2} & 6\le t\end{cases}$$

or:

$$y(t)=\begin{cases}\frac{1}{2}\left(t+\sin(t)\right) & 0\le t<6\\ 3+\cos(3)\sin(t-3) & 6\le t\end{cases}$$