Math 420, Spring 2023
Random Graphs: Second Team Homework

Exercise 1. [4pts] Write a function that computes the second smallest eigenvalue λ_1 of the unweighted normalized graph Laplacian for a given graph with n vertices. Then write a script that uses the assigned dateset, and computes the sequence of second smallest eigenvalue $\lambda_1(k)$ of the cumulative graph $Edges(1:k, 1:2)$, where $1 \leq k \leq m$ denotes the running number of edges. Specifically, order the edges according to their weights from file *weight.txt, starting with the largest weight first and then continue in a monotonic decreasing order. You may find easier to firstly create a data file, say graph.dat, from the raw data file assigned to your homework, that lists the edges in the appropriate order, and has the following format:

```
First line: n m
Second line: Edge1Vertex1 Edge1Vertex2
Third line: Edge2Vertex1 Edge2Vertex2
...
m+1st line: EdgemVertex1 EdgemVertex2
```

Then construct a sequence of unweighted graphs by adding one edge at a time in the order indicated before. For each graph in the sequence compute the unweighted normalized graph Laplacian and then λ_1. Denote by $\lambda_1(k)$ the second smallest eigenvalue of the unweighted normalized graph Laplacian corresponding to k edges. Thus obtain the sequence $(\lambda_1(1), \lambda_1(2), \ldots, \lambda_1(m))$.

1. Plot $\lambda_1 = \lambda_1(k)$, for $1 \leq k \leq m$.

2. Plot $1 - \lambda_1(k)$ as function of k, for $1 \leq k \leq m$.

3. Estimate the exponent α of the decay $1 - \lambda_1(k) = \frac{C}{k^\alpha}$. Specifically, find estimates for c_0 and α by least-squares fitting in the log-log model,

$$log(1 - \lambda_1(k)) = c_0 - \alpha \, log(k).$$

   Discard the first and the last 5 values of k. In other words, compute the log-log and fit the data for $6 \leq k \leq m - 5$. Compare the value of your estimated α to $\frac{1}{2}$ predicted by the result I quoted in class for the class of Erdös-Renyi random graphs.

Exercise 2. [6pts] Consider the weighted undirected graph inserted below.

1. [2pts] Write down the weight matrix W, the weighted graph Laplacian $\Delta = D - W$, and the normalized weighted graph Laplacian $\tilde{\Delta}$. Compute its eigenvalues and eigenvectors.

2. [2pts] Write a function that computes the Cheeger constant and the optimal partition for a given weight matrix W, and apply it to this graph. Determine both the optimal partition and the Cheeger constant h_G.

3. [2pts] Use the second smallest eigenpair obtained at the first part to determine an alternate partition (what we called in class the "initialization"). Find the value of the criterion minimized by the Cheeger constant and compare it to h_G.

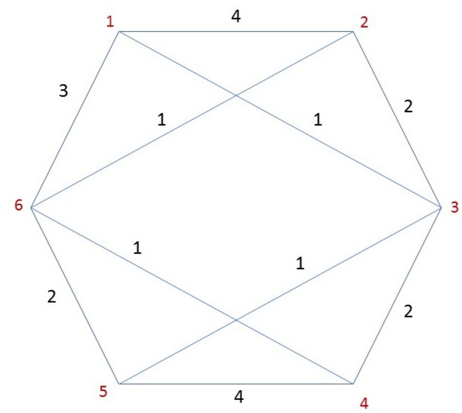

Figure 1: A weighted graph with $n = 6$ vertices. The vertex labels are marked in red. The edge weights are in black.