Lecture: Optimizations and Matrix Analysis

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March 2, 2023

Convex Sets. Convex Functions

A set $S \subset \mathbb{R}^n$ is called a *convex set* if for any points $x, y \in S$ the line segment $[x, y] := \{tx + (1-t)y \ , \ 0 \leq t \leq 1\}$ is included in S, $[x, y] \subset S$.

A function $f : S \to \mathbb{R}$ is called *convex* if for any $x, y \in S$ and $0 \leq t \leq 1$, $f(t\,x + (1-t)y) \leq t\,f(x) + (1-t)f(y)$.
Here S is supposed to be a convex set in $\mathbb{R}^n$.
Equivalently, f is convex if its epigraph is a convex set in $\mathbb{R}^{n+1}$. Epigraph: $\{(x, u) \ ; \ x \in S, u \geq f(x)\}$.

A function $f : S \to \mathbb{R}$ is called *strictly convex* if for any $x \neq y \in S$ and $0 < t < 1$, $f(t\,x + (1-t)y) < t\,f(x) + (1-t)f(y)$.

Convex Optimization Problems

The general form of a convex optimization problem:

$$\min_{x\in S} f(x)$$

where S is a closed convex set, and f is a convex function on S.
Properties:

1. Any local minimum is a global minimum. The set of minimizers is a convex subset of S.
2. If f is strictly convex, then the minimizer is unique: there is only one local minimizer.

In general S is defined by equality and inequality constraints: $S = \{g_i(x) \leq 0\ ,\ 1 \leq i \leq p\} \cap \{h_j(x) = 0\ ,\ 1 \leq j \leq m\}$. Typically h_j are required to be affine: $h_j(x) = a^T x + b$.

Primal-Dual Problems

Consider the *primal optimization problem*:

$$p^* = \quad \text{minimize} \quad f_0(x)$$
$$\textit{subject to}$$
$$f_i(x) \leq 0 \;,\; i \in [m]$$
$$h_j(x) = 0 \;,\; j \in [p]$$

Its associated *dual* problem is constructed by computing first the *Lagrange dual function* (known also as *dual function*):

$$g(\lambda, \mu) = \inf_{x \in Dom} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \nu_j h_j(x) \right)$$

and the *dual optimization problem*:

$$d^* = \quad \text{maximize} \quad g(\lambda, \nu)$$
$$\textit{subject to}$$
$$\lambda_i \geq 0 \;,\; i \in [m]$$
$$\nu_j \in \mathbb{R} \;,\; j \in [p]$$

Primal-Dual Problems (2)

Regardless of whether the primal problem is convex or not, always:

$$d^* \leq p^*$$

Hence the dual problem provides a lower bound of the optimum objective function. An obvious upper bound is given by $f_0(x_f)$ for any *feasible* x, i.e., one that satisfies the constraints $f_i(x_f) \leq 0$ and $h_j(x_f) = 0$.
When $d^* = p^*$ we say that *strong duality* holds. Some conditions (Slater's constraint qualification) guarantee strong duality.

Convex Programs

The hiarchy of convex optimization problems:

1. Linear Programs: Linear criterion with linear constraints
2. Quadratic Programs: Quadratic Criterion with Linear Constraints; Quadratically Constrained Quadratic Problems (QCQP); Second-Order Cone Program (SOCP)
3. Semi-Definite Programs(SDP)

Typical SDP:

$$\min_{\substack{X = X^T \geq 0 \\ trace(XB_k) = y_k \ ,\ 1 \leq k \leq p \\ trace(XC_j) \leq z_j \ ,\ 1 \leq j \leq m}} trace(XA)$$

CVX

Matlab package

Downloadable from: http://cvxr.com/cvx/ . Follows "Disciplined" Convex Programming – à la Boyd [1].

```
m = 20; n = 10; p = 4;
A = randn(m,n); b = randn(m,1);
C = randn(p,n); d = randn(p,1); e = rand;
cvx_begin
    variable x(n);
    minimize( norm( A * x - b, 2 ) )
    subject to
        C * x == d;
        norm( x, Inf ) <= e;
cvx_end
```

$$\begin{array}{ll} \min & \|Ax - b\| \\ Cx = d & \\ \|x\|_\infty \leq e & \end{array}$$

CVX

SDP Example

```
n = 10;
E1 = randn(n,n); d1 = randn(n,1);
E2 = randn(n,n); d2 = randn(n,1);
epsx = 1e-1;
cvx_begin sdp

    variable X(n,n) semidefinite;
    minimize trace(X);
    subject to
    X*ones(n,1) == zeros(n,1);
    abs(trace(E1*X)-d1)<=epsx;
    abs(trace(E2*X)-d2)<=epsx;

cvx_end
```

minimize $trace(X)$

subject to $X = X^T \geq 0$

$X \cdot 1 = 0$

$|trace(E_1 X) - d_1| \leq \varepsilon$

$|trace(E_2 X) - d_2| \leq \varepsilon$

References

S. Boyd, L. Vandenberghe, **Convex Optimization**, available online at: http://stanford.edu/ boyd/cvxbook/