Math 420, Spring 2023
Random Graphs: Second Team Homework

Exercise 1. [4pts] Write a function that computes the second smallest eigenvalue λ_1 of the unweighted normalized graph Laplacian for a given graph with n vertices. Then write a script that uses the assigned dateset, and computes the sequence of second smallest eigenvalue $\lambda_1(k)$ of the cumulative graph $Edges(1:k, 1:2)$, where $1 \leq k \leq m$ denotes the running number of edges. Specifically, order the edges according to their weights from file *weight.txt, starting with the largest weight first and then continue in a monotonic decreasing order. You may find easier to firstly create a data file, say graph.dat, from the raw data file assigned to your homework, that lists the edges in the appropriate order, and has the following format:

```
First line: n m
Second line: Edge1Vertex1 Edge1Vertex2
Third line: Edge2Vertex1 Edge2Vertex2
...
m+1st line: EdgemVertex1 EdgemVertex2
```

Then construct a sequence of unweighted graphs by adding one edge at a time in the order indicated before. For each graph in the sequence compute the unweighted normalized graph Laplacian and then λ_1. Denote by $\lambda_1(k)$ the second smallest eigenvalue of the unweighted normalized graph Laplacian corresponding to k edges. Thus obtain the sequence $(\lambda_1(1), \lambda_1(2), \ldots, \lambda_1(m))$.

1. Plot $\lambda_1 = \lambda_1(k)$, for $1 \leq k \leq m$.

2. Plot $1 - \lambda_1(k)$ as function of k, for $1 \leq k \leq m$.

3. Estimate the exponent α of the decay $1 - \lambda_1(k) = \frac{C}{k^\alpha}$. Specifically, find estimates for c_0 and α by least-squares fitting in the log-log model,

$$log(1 - \lambda_1(k)) = c_0 - \alpha \, log(k).$$

   Discard the first and the last 5 values of k. In other words, compute the log-log and fit the data for $6 \leq k \leq m - 5$. Compare the value of your estimated α to $\frac{1}{2}$ predicted by the result I quoted in class for the class of Erdös-Renyi random graphs.

Exercise 2. [6pts] Consider the weighted undirected graph inserted below.

1. [2pts] Write down the weight matrix W, the weighted graph Laplacian $\Delta = D - W$, and the normalized weighted graph Laplacian $\tilde{\Delta}$. Compute its eigenvalues and eigenvectors.

2. [2pts] Write a function that computes the Cheeger constant and the optimal partition for a given weight matrix W, and apply it to this graph. Determine both the optimal partition $(S_{opt}, \bar{S_{opt}})$ and the Cheeger constant h_G.

3. [2pts] Use the second smallest eigenpair obtained at the first part to determine an alternate partition (what we called in class the "initialization") $(S, \bar{S})$. Compute:

$$\frac{\sum_{x \in S, y \in \bar{S}} W(x,y)}{min(\sum_{x \in S} D(x,x), \sum_{x \in \bar{S}} D(x,x))}$$

and compare it to h_G.

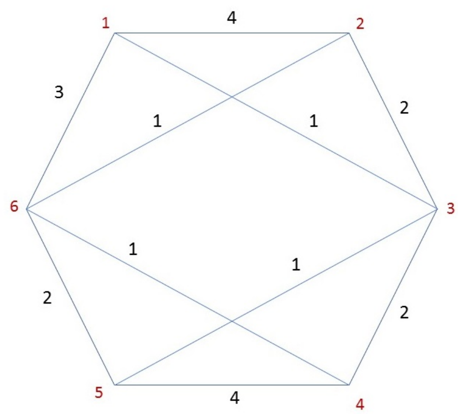

Figure 1: A weighted graph with $n = 6$ vertices. The vertex labels are marked in red. The edge weights are in black.