Lecture 10: SEIR Models

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The Susceptible-Exposed-Infectious-Removed (SEIR) Model

The Susceptible-Exposed-Infectious-Removed (SEIR) model is obtained from SIR by introducing a compartment between Susceptible and Infected of population that has been exposed to the virus but are not yet contagious:

$$\left\{\begin{array}{rcl} \frac{dS}{dt} & = & -\beta\frac{SI}{N} \quad , \quad S(0) \\ \frac{dE}{dt} & = & \beta\frac{SI}{N} - \delta E \quad , \quad E(0) \\ \frac{dI}{dt} & = & \delta E - \alpha I \quad , \quad I(0) \\ \frac{dR}{dt} & = & \alpha I \quad , \quad R(0) \end{array}\right. \qquad \text{(SEIR Model)}$$

where $\delta \geq 0$ is the rate of transition from exposed to infected. Its reciprocal $1/\delta$ represents the *average incubation period*.

If data is selected from the onset of infections, a natural initial condition is: $R(0) = N \gg I(0)$, $R(0) = 0$. The initial exposed population $E(0)$ may be set to $I(0)$, or can be fine tuned to fit the data. Assumin $E(0) = I(0)$ is known, the parameters that need to be calibrated are: $\alpha, \beta, \gamma, \delta, N$.

Simulations

Here are a few numerical results with the Euler's scheme:

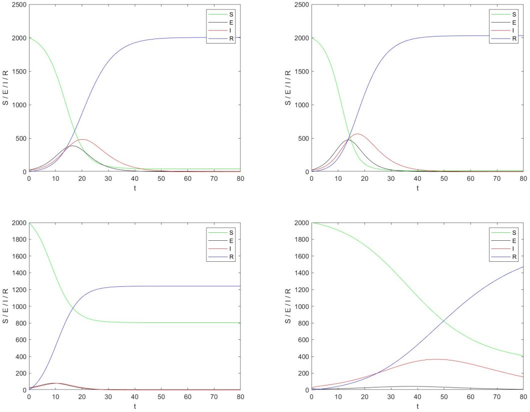


Figure: Top left: $\alpha = 0.2$, $\beta = 0.8$, $\delta = 0.3$; Top right: $\alpha = 0.2$, $\beta = 1.0$, $\delta = 0.3$; Bottom left: $\alpha = 1.0$, $\beta = 1.5$, $\delta = 1.0$; Bottom right: $\alpha = 0.08$, $\beta = 0.176$, $\delta = 0.8$

Agent Based Simulation

Let T_0 denote the Markov-Chain simulation time step. As in the case of SIR, the SEIR model can be implemented a piecewise linear differential system by replacing $I(t)$ in $\beta\frac{SI}{N}$ with the previous estimate $I((p-1)T_0)$ for the interval $t \in [(p-1)T_0, pT_0]$. The dynamical system becomes:

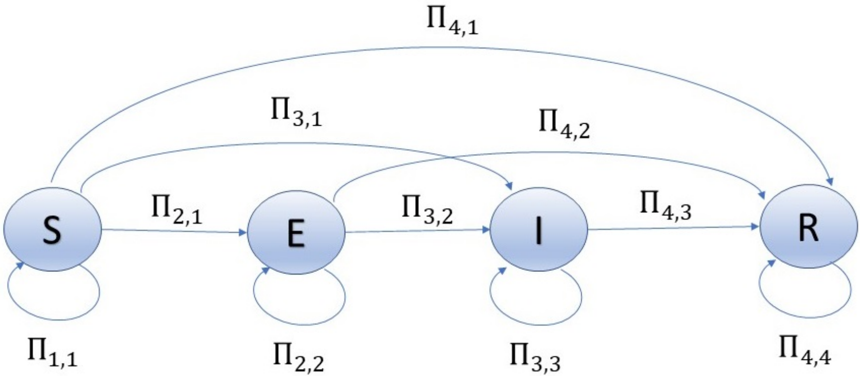
$$\left\{\begin{array}{rcl} \frac{dS(t)}{dt} & = & -\beta\frac{I((p-1)T_0)}{N}S(t) \quad , \quad S(0) \\ \frac{dE(t)}{dt} & = & \beta\frac{I((p-1)T_0)}{N}S(t) - \delta E(t) \quad , \quad E(0) \\ \frac{dI(t)}{dt} & = & \delta E(t) - \alpha I(t) \quad , \quad I(0) \\ \frac{dR(t)}{dt} & = & \alpha I(t) \quad , \quad R(0) \end{array}\right. \quad \text{for } (p-1)T_0 \leq t \leq pT_0$$

This defines the transition probability matrix:

$$\Pi = exp\left(\begin{bmatrix} T_0 - \frac{\beta}{N}I((p-1)T_0) & 0 & 0 & 0 \\ \frac{\beta}{N}I((p-1)T_0) & -\delta & 0 & 0 \\ 0 & \delta & -\alpha & 0 \\ 0 & 0 & \alpha & 0 \end{bmatrix}\right)$$

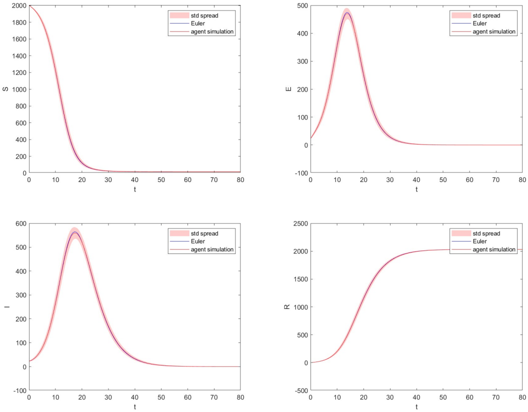
Agent Based Simulation (2)

The Markov Chain has the following form:



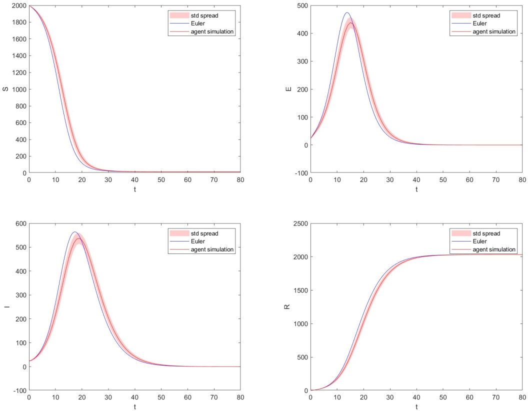
Examples of Agent Based Simulations

Here are results of a large number of simulations (10^3) for $T_0 = 0.01$, with parameters $\beta = 1$ and $\alpha = 0.2$, $\delta = 0.3$, initial condition $S_0 = 2000$, $E_0 = I_0 = 23$ and $R_0 = 0$. The numerical solution is obtained with the Euler scheme and a stepsize $h = 0.01$. The shaded area has semiwidth of one std of simulations



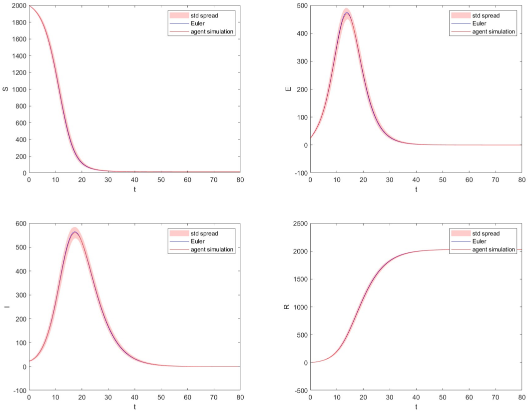
Examples of Agent Based Simulations

Here are results of a large number of simulations (10^3) for $T_0 = 1.00$, with parameters $\beta = 1$ and $\alpha = 0.2$, $\delta = 0.3$, initial condition $S_0 = 2000$, $E_0 = I_0 = 23$ and $R_0 = 0$. The numerical solution is obtained with the Euler scheme and a stepsize $h = 0.01$. The shaded area has semiwidth of one std of simulations



Examples of Agent Based Simulations

Here are results of a large number of simulations (10^3) for $T_0 = 0.001$, with parameters $\beta = 1$ and $\alpha = 0.2$, $\delta = 0.3$, initial condition $S_0 = 2000$, $E_0 = I_0 = 23$ and $R_0 = 0$. The numerical solution is obtained with the Euler scheme and a stepsize $h = 0.01$. The shaded area has semiwidth of one std of simulations



SEIR Model Calibration

Recall the model:

$$\left\{\begin{array}{lcl} \frac{dS}{dt} & = & -\beta\frac{SI}{N} \quad , \quad S(0) \\ \frac{dE}{dt} & = & \beta\frac{SI}{N} - \delta E \quad , \quad E(0) \\ \frac{dI}{dt} & = & \delta E - \alpha I \quad , \quad I(0) \\ \frac{dR}{dt} & = & \alpha I \quad , \quad R(0) \end{array}\right.$$

with the sub-compartments $X(t) = (1-\gamma)R(t)$ for "recovered" and $Y(t) = \gamma R(t)$ for deaths.
Before making useful predictions (testing), the model has to be calibrated.
We use the same calibration strategy as in the SIR model.

Optimization based on Forward Model Simulation

Pre-processing steps

Step 1: t_0. First detect the onset of infections, and reset the time origin to match this starting time t_0.

Step 2: $I(t)$. The time series of cumulative detected infections $\{V(0), \cdots, V(T_{T_{max}})\}$ should be turned into a rate of infections. The daily rate may not be relevant to infection transmissions. Instead use a time window to convert the cumulative count into a rate:

$$I(t) = V(t + t_0 + \tau_0) - V(t + t_0 - \tau_0) \quad , \quad t = 0, 1, 2, \cdots, T_{max}$$

where $\tau_0 > 0$ is chosed so that τ_0 accounts for average infection period. For Covid-19 it is somewhere between 5 days and 10 days. A possible value is $\tau_0 = 7$. Thus $I(t)$ measures the number of infections in a 2-week period centered around t.

Step 3: $Y_{measured}$. Align the time series of cumulative death with the onset time t_0:

$$Y_{measured}(t) = Y(t + t_0) \quad , \quad t = 0, 1, 2, \cdots, T_{max}.$$

Optimization based on Forward Model Simulation

The Meta Loop

A natural choice for initial conditions is given by: $S(0) = N$ and $R(0) = 0$, where we assumed $E(0) + I(0) \ll N$. Another choise is $E(0) = I(0)$ since we do not know the undetected number of infections at time 0 (perhaps $E(0) > I(0)$ is closer to the truth):

$$\left\{ \begin{array}{ccl} \frac{dS}{dt} & = & -\beta S \frac{I}{N} \quad , \quad S(0) = N \\ \frac{dE}{dt} & = & \beta \frac{SI}{N} - \delta E \quad , \quad E(0) \\ \frac{dI}{dt} & = & \delta E - \alpha I \quad , \quad I(0) \\ \frac{dR}{dt} & = & \alpha I \quad , \quad R(0) = 0 \quad , \quad Y = \gamma R \end{array} \right.$$

At this point, the parameters that need to be estimated are: $\{\alpha, \beta, \delta, \gamma, \rho\}$.
The Forward Model based calibration works like this:
1. Construct a search set Ω of the "free" parameters (α, β, δ)
2. For each $(\alpha, \beta, \delta) \in \Omega$: (i) run a SIR simulator using the Euler scheme that produces $(S_{sim}, E_{sim}, I_{sim}, R_{sim})$. (ii) Fit $\hat{\gamma}$ to match the observed time series $Y_{measured}$. (iii) Fit $\hat{\rho}$ that accounts the undercounting of actual infections in the observed time series I. (iv) Compute the value of the objective function $J_p(\alpha, \beta, \delta, \hat{\gamma}, \hat{\rho})$.
3. Select $(\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}, \hat{\rho})$ that minimize J.

Optimization based on Forward Model Simulation

Details: The objective function

The objective function J_p can be chosen to measure a norm of residuals $I - \rho I_{sim}$ and $Y_{measured} - \gamma R_{sim}$.
To do so, you need to choose and fix meta-parameters p, c_I and c_Y. For $1 \leq p < \infty$ define

$$J_p(\alpha, \beta, \delta, \gamma, \rho) = c_I \sum_{t=0}^{T_{max}} |I(t) - \rho I_{sim}(t)|^p + c_Y \sum_{t=0}^{T_{max}} |Y_{measured}(t) - \gamma R_{sim}(t)|^p$$

For $p = \infty$ define

$$J_\infty(\alpha, \beta, \delta, \gamma, \rho) = c_I \max_{0 \leq t \leq T_{max}} |I(t) - \rho I_{sim}(t)| + c_Y \max_{0 \leq t \leq T_{max}} |Y_{measured}(t) - \gamma R_{s}$$

Typical choices: $p = 1, 2, \infty$ and $(c_I, c_Y) = (0, 1)$ (if the cumulative detected infections are unreliable) or $(c_I, c_Y) = (1, 1)$ (if detected infections is a reliable measure).

The Calibration Algorithm for SEIR Models

Algorithm (Meta-Algorithm for SEIR Calibration)

Inputs: *Time series* $\{V(0), \cdots, V(T)\}$, $\{Y(0), \cdots, Y(T)\}$. *Parameters:* V_{min} *(default,* $V_{min} = 5$*),* τ_0 *(default,* $\tau_0 = 7$*),* N *= Population,* $E(0)$ *(default* $E(0) = I(0)$*),* $p \in [1, \infty]$ *(default,* $p = 2$*),* $c_I, c_Y > 0$*. Search set* Ω*.*

1. *Detect the onset of the infection* t_0 *as the first time so that* $V(t_0) \geq V_{min}$*, reset the time origin, and create the time series of infection rates* $I(t) = V(t + t_0 + \tau_0) - V(t + t_0 - \tau_0)$*, and aligned cumulative death* $Y_{measured}(t) = Y(t + t_0)$, $0 \leq t \leq T_{max}$.
2. *For each* $(\alpha, \beta, \delta) \in \Omega$ *repeat:*
    1. *Simulate a SEIR model with parameters* (α, β, δ) *and initial condition* $S_{sim}(0) = N$, $E_{sim}(0) = E(0)$, $I_{sim}(0) = I(0)$, $R_{sim}(0) = 0$*, and obtain daily time series* $(S_{sim}, E_{sim}, I_{sim}, R_{sim})$.
    2. *Solve* $\hat{\gamma} = argmin_{\gamma} \| Y_{measured} - \gamma R_{sim} \|_p$.
    3. *Solve* $\hat{\rho} = argmin_{\rho} \| I - \rho I_{sim} \|_p$.
    4. *Compute the objective function* $J = J_p(\alpha, \beta, \delta, \hat{\gamma}, \hat{\rho})$.
3. *Determine the minimum and the minimizer of* J.

Outputs: *Estimated* $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}, \hat{\rho}$ *and minimum value of the objective function* J_{min}.

Numerical results (1)

Analysis of DC data for 2020

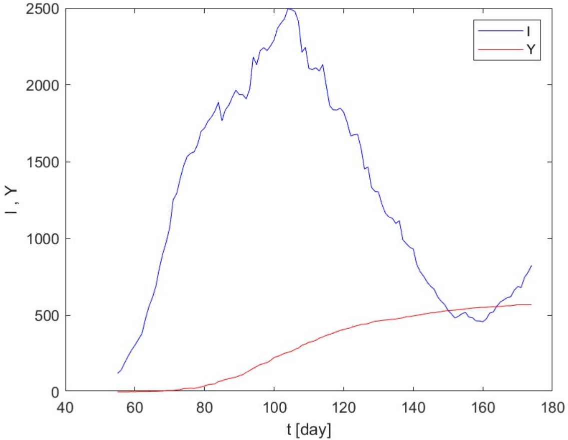
$V_{min} = 5$, $\tau_0 = 7$ [days]. Onset $t_0 = 55$ (Monday: March 16, 2020), $T_{max} = 119$, $N = 705749$.

Notable dates:

Local dips: day=152 (Sunday: 6/21/2020), day=159 (Sunday: 6/28/2020).

Memorial weekend: Sunday May 24, 2020: day = 124 .

4th of July: Saturday July 4, 2020: day = 165.



Numerical results (2)

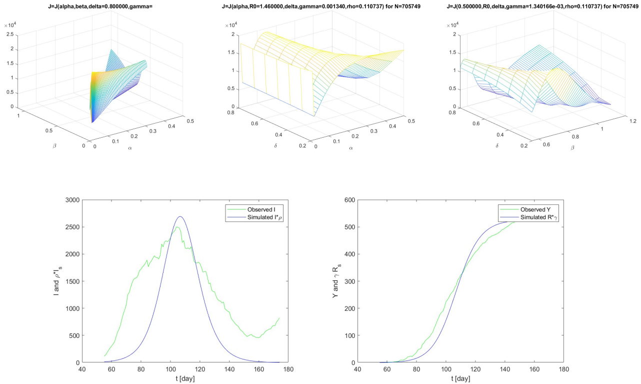
Analysis of DC data for 2020. $p = 1$, $c_I = 0$, $c_Y = 1$

Results:
$\hat{\alpha} = 0.5$, $\delta = 0.8$,

$\hat{\beta} = 0.73$,
$\hat{R}_0 = 1.46$,

$\hat{\gamma} = 0.134\%$.

$\hat{\rho} = 11\%$
$J_{optim} = 3103$.



Numerical results (3)

Analysis of DC data for 2020. $p = 2$, $c_I = 0$, $c_Y = 1$

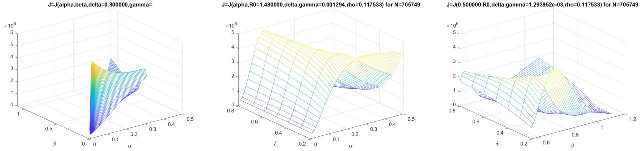
Results:
$\hat{\alpha} = 0.5$, $\delta = 0.8$,

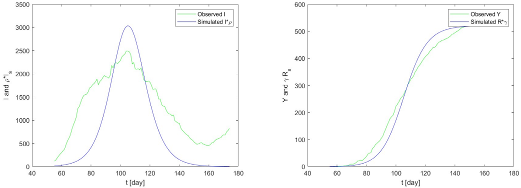
$\hat{\beta} = 0.74$, $\hat{R}_0 = 1.48$,

$\hat{\gamma} = 0.13\%$.

$\hat{\rho} = 11.75\%$.

$J_{optim} = 117406$.





Numerical results (4)

Analysis of DC data for 2020. $p=\infty$, $c_I=0$, $c_Y=1$

Results:

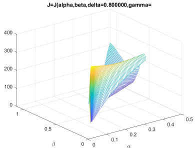
$\hat{\alpha}=0.5$, $\delta=0.8$,

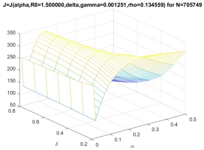
$\hat{\beta}=0.75$, $\hat{R}_0=1.5$,

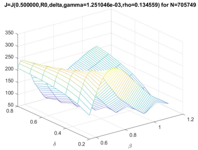
$\hat{\gamma}=0.125\%$.

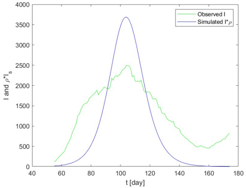
$\hat{\rho}=13.45\%$.

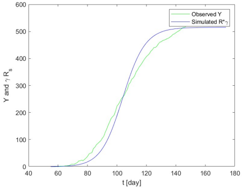
$J_{optim}=52.7$.











Numerical results (5)

Analysis of DC data for 2020. $p = 1$, $c_I = 1$, $c_Y = 1$

Results:

$\hat{\alpha} = 0.5$, $\delta = 0.8$,

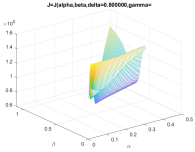
$\hat{\beta} = 0.71$,

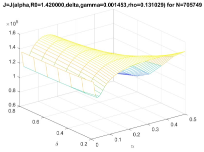
$\hat{R}_0 = 1.42$,

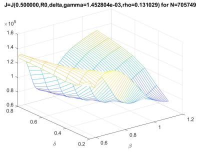
$\hat{\gamma} = 0.146\%$.

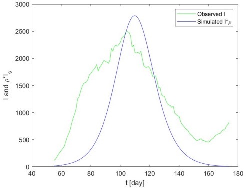
$\hat{\rho} = 13.1\%$

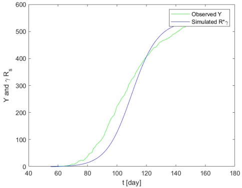
$J_{optim} = 74151$.











Numerical results (6)

Analysis of DC data for 2020. $p = 2$, $c_I = 1$, $c_Y = 1$

Results:

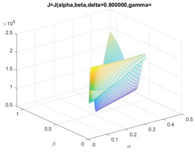
$\hat{\alpha} = 0.5$, $\delta = 0.8$,

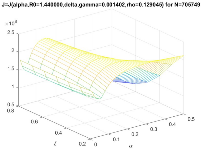
$\hat{\beta} = 0.72$,
$\hat{R}_0 = 1.44$,

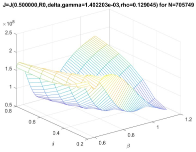
$\hat{\gamma} = 0.14\%$.

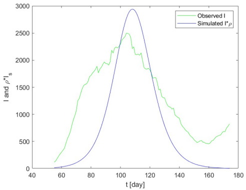
$\hat{\rho} = 12.9\%$.

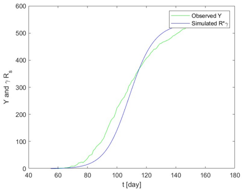
$J_{optim} = 57030309$.











Numerical results (7)

Analysis of DC data for 2020. $p = \infty$, $c_I = 1$, $c_Y = 1$

Results:
$\hat{\alpha} = 0.5$, $\delta = 0.8$,

$\hat{\beta} = 0.75$, $\hat{R}_0 = 1.5$,

$\hat{\gamma} = 0.125\%$.

$\hat{\rho} = 13.45\%$.

$J_{optim} = 1308$.