

I.

1. Use the Fourier transform of $g(x) = e^{-|x|}$, $G(s) = \frac{2}{1+4\pi^2 s^2}$ and the affine change of coordinate $y = 2x + 5$,

$$F(s) = \int e^{-2\pi i s x} e^{-|2x+5|} dx = \frac{1}{2} \int e^{-2\pi i s (y-5)/2} e^{-|y|} dy = \frac{e^{5\pi i s}}{1 + \pi^2 s^2}$$

2. Use the dilation rule $y = \sqrt{\frac{3}{\pi}}x$ for the Fourier transform $\gamma(y) = e^{-\pi y^2} \mapsto \Gamma(s) = e^{-\pi s^2}$

$$F(s) = \int e^{-2\pi i s x} e^{-3x^2} dx = \sqrt{\frac{\pi}{3}} e^{-\pi^2 s^2/3}$$

3. Notice $f(x)$ is the inverse Fourier transform of e^{-s^4} . Hence

$$F(s) = e^{-s^4}$$

4. Complete the square and use the Fourier transform of $g(x) = \frac{1}{1+x^2}$ is $G(s) = \pi e^{-2\pi|s|}$

$$F(s) = \int e^{-2\pi i s x} \frac{2}{(x-2)^2 + 1} dx = 2\pi e^{-4\pi i s - 2\pi|s|}$$

5. Use the following sequence

$$f_1(x) = \frac{1}{1+x^2} \mapsto F_1(s) = \pi e^{-2\pi|s|} \quad (1)$$

$$f_2(x) = \frac{x}{1+x^2} = x f_1(x) \mapsto F_2(s) = -\frac{1}{2\pi i} \frac{d}{ds} F_1(s) = -i\pi \operatorname{sign}(s) e^{-2\pi|s|} \quad (2)$$

$$f(x) = \frac{2(x-2)}{(x-2)^2 + 1} = 2f_2(x-2) \mapsto F(s) = -2i\pi \operatorname{sign}(s) e^{-4\pi i s - 2\pi|s|} \quad (3)$$

where $\operatorname{sign}(s) = 1$ for $s > 0$ and $\operatorname{sign}(s) = -1$ for $s < 0$. At $s = 0$,

$$F(0) = \int_{-\infty}^{\infty} f(x) dx = 2 \int_{-\infty}^{\infty} \frac{x-2}{(x-2)^2 + 1} dx = 2 \lim_{R \rightarrow \infty} \int_{-R}^R \frac{y}{y^2 + 1} dy = 0$$

(the last integral is zero because $y/(y^2+1)$ is an odd function). Thus we get $F(s) = -2i\pi \operatorname{sign}(s) e^{-4\pi i s - 2\pi|s|}$ where $\operatorname{sign}(0) = 0$.

Note. As an alternative to computing $F(0)$ explicitly, one may invoke the Dirichlet theorem of pointwise convergence and regularize F at 0 by $F(0) = 0.5(F(0-) + F(0+))$.

II.

6. We look for a function $f : \mathbb{R} \rightarrow \mathbb{C}$ that satisfies the given equation. Since we have the freedom to impose any condition we want, we make a choice $f(-x) = f(x)$ (that is, we look for an even function). Then its Fourier transform is:

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} e^{-2\pi i s x} f(x) dx = \int_0^{\infty} (e^{-2\pi i s x} f(x) + e^{2\pi i s x} f(-x)) dx \\ &= \int_0^{\infty} (e^{-2\pi i s x} + e^{2\pi i s x}) f(x) dx = 2 \int_0^{\infty} \cos(2\pi s x) f(x) dx \end{aligned}$$

Note also:

$$F(-s) = F(s)$$

We want to have $F(s) = 2e^{-s}$ for $s > 0$. Then, because of the symmetry $F(-s) = F(s)$, we obtain $F(s) = 2e^{-|s|}$. The function f is now obtained through the synthesis equation (the inverse Fourier transform):

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi isx} F(s) ds = 2 \int_{-\infty}^{\infty} e^{2\pi isx} e^{-|s|} ds = \frac{4}{1 + 4\pi^2 x^2}$$

Note that indeed $f(-x) = f(x)$ as we assumed.

7. We are asked to find a function $F : \mathbb{R} \rightarrow \mathbb{C}$ that satisfies the given equation. We choose f to be odd (anti-symmetric), that is $f(-x) = -f(x)$. Then:

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} e^{-2\pi isx} f(x) dx = \\ &= \int_0^{\infty} (e^{-2\pi isx} f(x) + e^{2\pi isx} f(-x)) dx = -2i \int_0^{\infty} f(x) \sin(2\pi sx) dx \end{aligned}$$

Note, by direct computation, that:

$$F(-s) = 2i \int_0^{\infty} f(x) \sin(2\pi sx) dx = -F(s)$$

Thus the Fourier transform F must satisfy:

$$F(s) = \begin{cases} -2i & , \quad 0 < s < 1 \\ 2i & , \quad -1 < s < 0 \\ 0 & , \quad |s| > 1 \end{cases}$$

(and the regularization at -1,0,1). The function f is obtained by computing the inverse Fourier transform:

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi isx} F(s) ds = 2 \frac{1 - \cos(\pi x)}{\pi x}$$

Note $f(-x) = -f(x)$ as assumed.

8. Note the left hand side is the Fourier transform of f at $-x/(2\pi)$. Hence

$$F\left(-\frac{x}{2\pi}\right) = e^{-\pi x^2}$$

Thus

$$F(s) = e^{-\pi(2\pi s)^2}$$

The inverse Fourier transform gives

$$f(x) = \frac{1}{2\pi} e^{-\pi(x/(2\pi))^2} = \frac{1}{2\pi} e^{-\frac{x^2}{4\pi}}$$

III.

9. Applying Fourier transform on both sides we get:

$$-(2\pi is)^2 F(s) + F(s) = G(s)$$

where $G(s) = \frac{1}{1 + \pi^2 s^2}$. Hence

$$F(s) = \frac{G(s)}{1 + 4\pi^2 s^2} = \frac{1}{(1 + 4\pi^2 s^2)(1 + \pi^2 s^2)}$$

10. Note

$$F(s) = \frac{4/3}{1 + 4\pi^2 s^2} - \frac{1/3}{1 + \pi^2 s^2}$$

Hence

$$f(x) = \frac{2}{3} e^{-|x|} - \frac{1}{3} e^{-2|x|}$$