L12

Example of an integral equation

Consider the integral equation: $f(x) = \int_{-\infty}^{\infty} f(u) \cdot \text{sinc}(x-u)\, du$

1) What can we infer about the Fourier transform of f?

2) Find a solution of this equation.

Solution:

Note that $\int_{-\infty}^{\infty} f(u) \cdot \text{sinc}(x-u)\, du = (f * \text{sinc})(x)$.

Equation: $f = f * \text{sinc}$ $(*)$

Let F denote the Fourier transform of f.

We know: $\widehat{\text{sinc}} = \Pi$.

Therefore ~~the equation (*) be~~ by applying Fourier transform on both sides of $(*)$ we get:

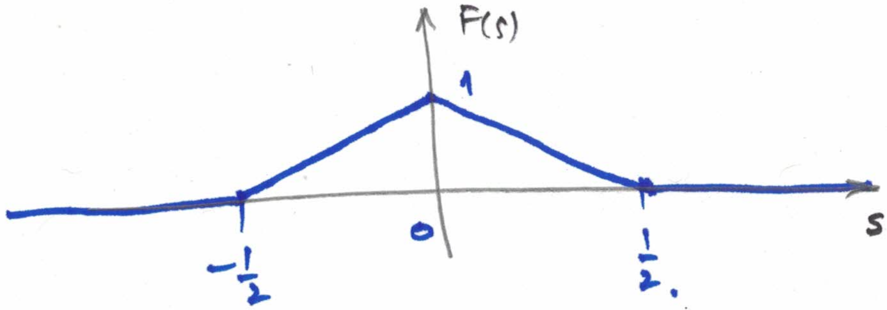
$$F(s) = F(s) \cdot \Pi(s) \qquad (**)$$

$\Rightarrow$ For $|s| \geq \frac{1}{2}$, $F(s) = 0$.

For $|s| < \frac{1}{2}$, $(**) \Rightarrow F(s) = F(s)$. $\longrightarrow$ Represents no contraint on F.

1 $\longrightarrow$ We infer that $F(s) = 0$, for $s \leq -\frac{1}{2}$ or $s \geq \frac{1}{2}$.

For $-\frac{1}{2} < s < \frac{1}{2}$, there is no contraint on $F(s)$.

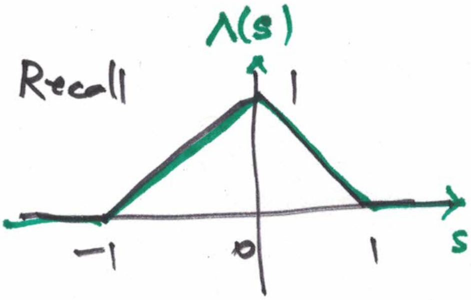


For ②

Choose :

$$F(s) = \begin{cases} 0, & |s| \geq \frac{1}{2} \\ 1 - 2|s|, & |s| < \frac{1}{2}. \end{cases}$$

Recall



$$\Lambda(s) = \begin{cases} 0, & |s| > 1 \\ 1 - |s|, & |s| \leq 1. \end{cases}$$

Therefore : $F(s) = \Lambda(2s)$

The function f is obtained by Fourier inversion :

Fourier Transform		Inverse Fourier Transform
$\Pi(s)$		$\mathrm{Sinc}(x)$
$\Lambda = \Pi * \Pi.$	$\xrightarrow{\text{Conv. Rule}}$	$\mathrm{Sinc}^2(x)$
$F(s) = \Lambda(2s).$	$\xrightarrow[\text{Rule.}]{\text{Scaling}}$	$f(x) = \frac{1}{2}\left(\mathrm{sinc}\left(\frac{x}{2}\right)\right)^2$

Hence $f(x) = \frac{1}{2}\left(\mathrm{sinc}\left(\frac{x}{2}\right)\right)^2$ is a solution of $f * \mathrm{sinc} = f$.

Relationship between Convolutions and Linear Filters

If $f, g : \mathbb{R} \to \mathbb{C}$ are 1-periodic functions, their (periodic) convolution:

$$f * g\,(x) = \int_0^1 f(x-y)\,g(y)\,dy.$$

On the other hand:

the Fourier Series expansion of f and g:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}, \quad c_n = \int_0^1 e^{-2\pi i n x} f(x)\,dx$$

$$g(x) = \sum_{n=-\infty}^{\infty} d_n e^{2\pi i n x}, \quad d_n = \int_0^1 e^{-2\pi i n x} g(x)\,dx.$$

Remark:

$f*g$ is also a 1-periodic function:

$$(f*g)(x+1) = \int_0^1 \underbrace{f(x+1-y)}_{f(x-y)}\,g(y)\,dy = \int_0^1 f(x-y)\,g(y)\,dy = (f*g)(x).$$

The Fourier series expansion of $f*g$:

$$f*g\,(x) = \sum_{n=-\infty}^{\infty} \gamma_n e^{2\pi i n x}, \quad \underline{\underline{\gamma_n}} = \int_0^1 e^{-2\pi i n x} (f*g)(x)\,dx$$

Question: How to compute γ_n directly from $(c_n)_n$, $(d_n)_n$?

$$\gamma_n = \int_0^1 e^{-2\pi i n x} (f * g)(x)\, dx = \int_0^1 e^{-2\pi i n x} \left(\int_0^1 f(x-y)\, g(y)\, dy \right) dx =$$

$$= \int_0^1 \int_0^1 \underbrace{e^{-2\pi i n (x-y+y)}}_{e^{-2\pi i n (x-y)} \cdot e^{-2\pi i y n}} f(x-y)\, g(y)\, dx\, dy =$$

$$= \int_0^1 e^{-2\pi i n y} g(y) \left(\int_0^1 e^{-2\pi i n (x-y)} f(x-y)\, dx \right) dy =$$

$$t = x-y, \quad dt = dx \longrightarrow \int_{-y}^{1-y} e^{-2\pi i n t} f(t)\, dt = \int_0^1 e^{-2\pi i n t} f(t) dt$$

$$= \int_0^1 e^{-2\pi i n y} g(y) \underbrace{\left(\int_0^1 e^{-2\pi i n t} f(t)\, dt \right)}_{c_n} dy = c_n \cdot \underbrace{\int_0^1 e^{-2\pi i n y} g(y)\, dy}_{d_n} =$$

$$= c_n \cdot d_n$$

Summary:

$$\gamma_n = c_n \cdot d_n.$$

n^{th} Fourier coeff of f (for c_n)

n^{th} Fourier coeff. of g (for d_n).

γ_n: n^{th} Fourier coeff of $f * g$.

$*$: 1-periodic convolution.

The converse is also true!

So far:

$$f \longleftrightarrow (c_n)_n.$$

$$g \longleftrightarrow (d_n)_n$$

$$f * g \longleftrightarrow (c_n \cdot d_n)_n.$$

Consider now: $c * d$, convolution over sequences of coefficients

If $c = (c_n)_{n \in \mathbb{Z}}$, $d = (d_n)_{n \in \mathbb{Z}}$

$$(c*d)_n = \sum_{k=-\infty}^{\infty} c_{n-k} \cdot d_k$$

Question: What is the 1-periodic function $h(x) = \sum_{n=-\infty}^{\infty} (c*d)_n \cdot e^{2\pi i n x}$?

$$h(x) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} c_{n-k} \cdot d_k \right) e^{2\pi i n x} =$$

$$= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{2\pi i (n-k+k)x} c_{n-k} \cdot d_k = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{2\pi i (n-k)x} c_{n-k} \cdot \underbrace{e^{2\pi i k x} d_k}$$

$$= \sum_k e^{2\pi i k x} d_k \cdot \sum_n e^{2\pi i (n-k)x} c_{n-k}.$$

$m = n-k.$

$$= \underbrace{\sum_k d_k e^{2\pi i k x}}_{g(x)} \underbrace{\sum_m c_m e^{2\pi i m x}}_{f(x).} = f(x) \cdot g(x).$$

Rule:

If $\quad f$: 1-periodic $\longleftrightarrow (c_n)_n.$

$\quad\ \ g$: 1-periodic $\longleftrightarrow (d_n)_n.$

Then: $\quad f * g \longleftrightarrow (c_n \cdot d_n)_n.$

$\quad\ \ f \cdot g \longleftrightarrow (c * d)_n.$

Linear Systems : <u>Linear Time Invariant (LTI) Systems</u>

An discrete-time LTI system :

Input: $(c_n)_{n\in\mathbb{Z}} \longrightarrow \boxed{\text{LTI}} \longrightarrow$ Output: $(d_n)_{n\in\mathbb{Z}}.$

LTI : takes an input sequence $(c_n)_n$ and returns an output sequence $(d_n)_n$

Linearity

such that :

① If. input is $(a\cdot c_n)_{n\in\mathbb{Z}}$ then output is $(a\cdot d_n)_{n\in\mathbb{Z}}$

for any $a\in\mathbb{C}$

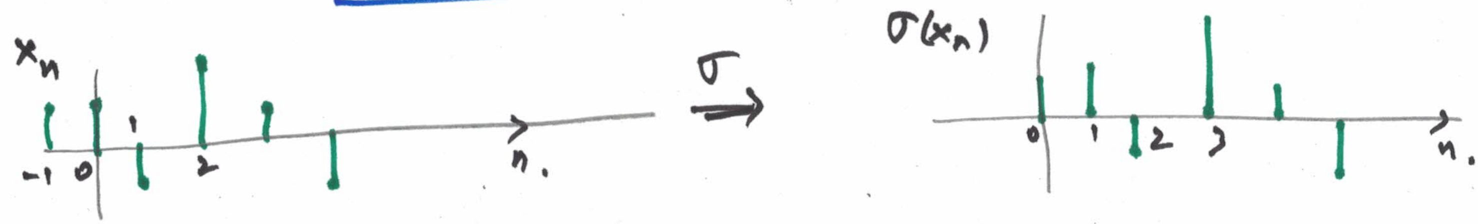
② If. $\text{input}_1 = (c_n)_n \longrightarrow \text{output}_1 = (d_n)_n.$

$\quad\ \ \text{input}_2 = (x_n)_n \longrightarrow \text{output}_2 = (y_n)_n.$

Then $\quad \text{input} = (c_n + x_n)_n \longrightarrow \text{output} = (d_n + y_n)_n.$

Let. $\sigma: x = (x_n)_{n\in\mathbb{Z}} \longrightarrow (\sigma(x))_n = x_{n-1}$.

denote the time-shift:



③ If input $c = (c_n)_{n\in\mathbb{Z}} \xrightarrow{LTI}$ output: $d = (d_n)_{n\in\mathbb{Z}}$

Then for input $\sigma(c) \xrightarrow{LTI}$ output: $\sigma(d)$.

Time-Invariance.

Remark: ③ $\Rightarrow$ If input $c = (c_n) \xrightarrow{LTI}$ output: $d = (d_n)_n$.

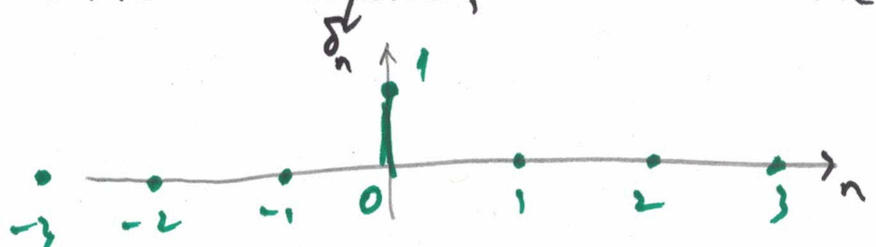
Then for any integer m,

input $\sigma^m(c) \xrightarrow{LTI}$ output: $\sigma^m(d)$.

$(\sigma^m(c))_n = c_{n-m}$. $\xrightarrow{LTI}$ output: $(\sigma^m(d))_n = d_{n-m}$

Let $\delta = (\delta_{n,0})_{n\in\mathbb{Z}}$, where $\delta_{n,0} = \begin{cases} 1, & n=0 \\ 0, & n\neq 0. \end{cases}$

be the unit sequence, also known as the impulse.



Note: $\left(\sigma^m(\delta)\right)_n = \delta_{n-m,0} = \delta_{n,m} = \begin{cases} 1, & n = m \\ 0, & n \neq m. \end{cases}$

Then:

$$x = (x_n)_{n\in\mathbb{Z}} = \underbrace{\left(\sum_{m=-\infty}^{\infty} x_m \cdot \sigma^m(\delta)\right)_n}$$

For a fixed n: $\sum_m x_m \cdot \left(\sigma^{(m)}(\delta)\right)_n = \sum_m x_m \cdot \delta_{n,m} = x_n$.

We showed:

$$x = \sum_{m=-\infty}^{\infty} x_m \cdot \sigma^m(\delta)$$

$\dashrightarrow$ $\{\sigma^m(\delta),\ m\in\mathbb{Z}\}$ is a basis for sequences.

$$x = \sum_m x_m \sigma^m(\delta) \longrightarrow \boxed{\text{LTI}} \longrightarrow y = ?$$

If $\delta \longrightarrow \boxed{\text{LTI}} \longrightarrow h.$ h denotes the output when input $= \delta$ (impulse).

$h =$ "impulse response".

Then:

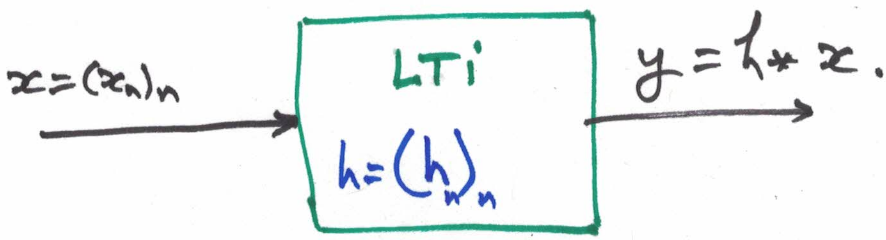
$$\sum_m x_m \cdot \sigma^m(\delta) \longrightarrow \boxed{\text{LTI}} \longrightarrow y = \sum_m x_m \cdot \sigma^m(h)$$

Therefore:

$$y_n = \left(\sum_m x_m \, \sigma^m(h)\right)_n = \sum_m x_m \cdot \left(\sigma^m(h)\right)_n =$$

$$= \sum_m x_m \cdot h_{n-m} = \sum_{m=-\infty}^{\infty} h_{n-m} \cdot x_m = (h * x)_n .$$

<u>Conclusion</u>: Any LTI is characterized by the following input-output relationship:



where $h = (h_n)_n$ is the impulse response.

A different characterization:

$$x = (x_n)_{n \in \mathbb{Z}} \longrightarrow X(\omega) = \sum_{n=-\infty}^{\infty} x_n e^{2\pi i n \omega}$$

$$y = (y_n)_{n \in \mathbb{Z}} \longrightarrow Y(\omega) = \sum_{n=-\infty}^{\infty} y_n e^{2\pi i n \omega}$$

$$h = (h_n)_n \longrightarrow H(\omega) = \sum_{n=-\infty}^{\infty} h_n e^{2\pi i n \omega} .$$

We obtain:

$$\boxed{Y(\omega) = H(\omega) \cdot X(\omega)}$$

↳ Transfer Function of LTI