Sampling of Band limited Functions (2)

Recall: If $f \in B^2_\Omega = \{ f \in L^2(\mathbb{R}) : \quad F(s) = 0, \ \forall |s| > \Omega \}$

then:

$$f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2\Omega}\right) \cdot \operatorname{sinc}(2\Omega x - n).$$

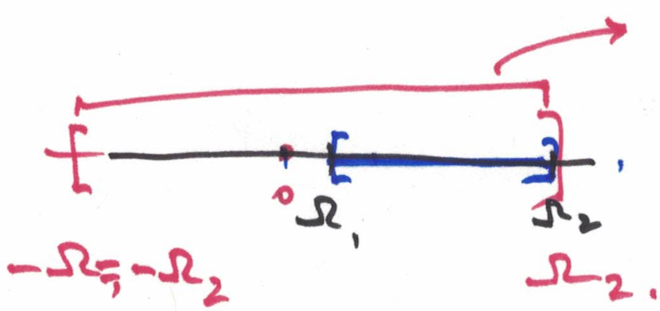
$\boxed{\text{Refinment 2}}$ Assume. $f : \mathbb{R} \to \mathbb{C}$, $f \in L^2(\mathbb{R})$: $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$

and. F is supported in $[\Omega_1, \Omega_2]$: $\quad F(s) = 0, \ \forall s < \Omega_1$ or $s > \Omega_2$.

Problem: How to recover/compute $f(x)$ from its samples.

If we know $\{ f(nT), \ n \in \mathbb{Z} \}$. for some $T > 0$

how to compute $f(x)$?



Solution: Let: $\Omega_0 = \dfrac{\Omega_1 + \Omega_2}{2}$ $\longrightarrow$ center frequency.

$T = \dfrac{1}{\Omega_2 - \Omega_1}$, $\Omega_2 - \Omega_1$: effective bandwidth

Then:

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} f(nT)\, e^{2\pi i \Omega_0 (x-nt)} \cdot \operatorname{sinc}\left(\frac{x-nT}{T}\right).}$$

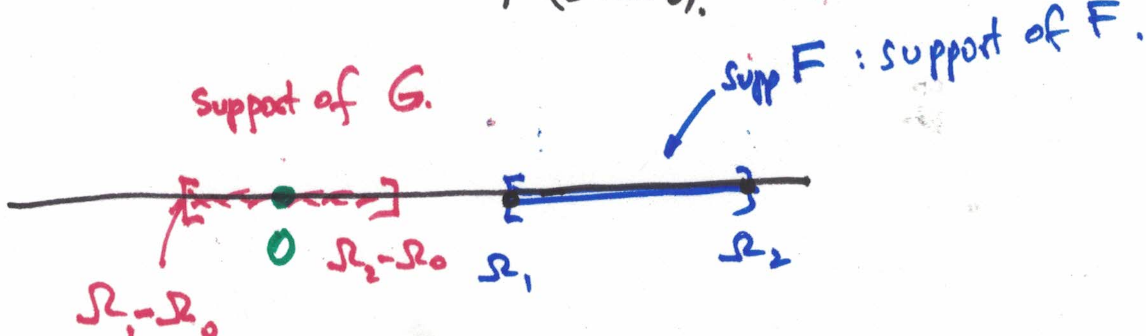
Why:

Set $g(x) = e^{-2\pi i \Omega_0 x} \cdot f(x)$.

Compute its Fourier transform:

$$G(s) = \int_{-\infty}^{\infty} e^{-2\pi i s x} g(x)\,dx = \int_{-\infty}^{\infty} e^{-2\pi i (s+\Omega_0)x} f(x)\,dx =$$

$$= F(s+\Omega_0).$$



$$\Omega_1 - \Omega_0 = \Omega_1 - \frac{\Omega_1+\Omega_2}{2} = \frac{\Omega_1 - \Omega_2}{2}$$

$$\Omega_2 - \Omega_0 = \Omega_2 - \frac{\Omega_1+\Omega_2}{2} = \frac{\Omega_2 - \Omega_1}{2}$$

$\mapsto \Omega_2 - \Omega_0 = -(\Omega_1 - \Omega$

Hence g is Ω-bandlimited, where $\Omega = \dfrac{\Omega_2 - \Omega_1}{2}$.

Shannon's Sampling formula to g:

$$g(x) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{\cancel{2} \cdot \frac{\Omega_2 - \Omega_1}{\cancel{2}}} \right) \cdot \operatorname{sinc}\left(\cancel{2} \frac{\Omega_2 - \Omega_1}{\cancel{2}} x - n \right)$$

$$T = \frac{1}{\Omega_2 - \Omega_1} \Rightarrow g(x) = \sum_{n=-\infty}^{\infty} g(nT) \operatorname{sinc}\left(\frac{x - nT}{T} \right).$$

$$e^{-2\pi i \Omega_0 x} \cdot f(x) = \sum_{n=-\infty}^{\infty} e^{-2\pi i \Omega_0 nT} f(nT) \cdot \operatorname{sinc}\left(\frac{x - nT}{T} \right).$$

$$\Rightarrow \quad f(x) = \sum_{n=-\infty}^{\infty} e^{2\pi i \Omega_0 (x - nT)} f(nT) \operatorname{sinc}\left(\frac{x - nT}{T} \right).$$

Terminology: $\sum_{n=-\infty}^{\infty} c_n \cdot \operatorname{sinc}\left(\frac{x - nT}{T} \right)$. are called cardinal series

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Additional Formula: Last time we showed: $\{ \sqrt{2\Omega} \operatorname{sinc}(2\Omega x - n), n \in \mathbb{Z} \}$ is ONB for B^2_Ω.

one consequence $\dashrightarrow$ $f \in B^2_\Omega$, $f = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n$: Shannon's formula

Another consequence: $\dashrightarrow$ $\|f\|^2 = \sum_{n=-\infty}^{\infty} |\langle f, e_n \rangle|^2$, Last time: $\langle f, e_n \rangle = \frac{1}{\sqrt{2\Omega}} f\left(\frac{n}{2\Omega}\right)$

Hence: $$\boxed{\forall f \in B^2_\Omega, \quad \int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\Omega} \sum_{n=-\infty}^{\infty} \left| f\left(\frac{n}{2T}\right) \right|^2 = T \sum_{n=-\infty}^{\infty} |f(nT)|^2}$$

Approximation by Cardinal Series

(see section 8.4). $\dashrightarrow$ (formula 55, and after).

The goal is to bound:

$$e_{N,M}(t) = f(t) - \sum_{n=M}^{N} f(nT) \cdot \operatorname{sinc}\left(\frac{t-nT}{T}\right).$$

[Fix $T>0$, and two integers M, N.

Assume $f : \mathbb{R} \to \mathbb{C}$ satisfies "some" technical conditions
(f may <u>not</u> be bandlimited):

(1) $\displaystyle\int_{-\infty}^{\infty} |F(s)|\, ds < \infty$ (small "tail"), $F \in L^1(\mathbb{R})$

(2) $\displaystyle\int_{-\infty}^{\infty} |f(t)|^2\, dt < \infty$, (finite energy), $f \in L^2(\mathbb{R})$

<u>Then</u>:

$\boxed{A}$ For every $t \in \mathbb{R}$,

$$\left| f(t) - \sum_{n=M}^{N} f(nT) \cdot \operatorname{sinc}\left(\frac{t-nT}{T}\right) \right| \leq \underbrace{2 \int_{|\omega| > \frac{1}{2T}} |F(\omega)|\, d\omega}_{\text{out-of-band. aliasing error.}} + \underbrace{\left(\sum_{n<M} |f(nT)|^2 \right)^{1/2} + \left(\sum_{n>N} |f(nT)|^2 \right)^{1/2}}_{\text{truncation error.}}$$

$$\int_{|\omega|>\frac{1}{2T}} |F(\omega)|\,d\omega = \int_{-\infty}^{-1/2T} |F(\omega)|\,d\omega + \int_{\frac{1}{2T}}^{\infty} |F(\omega)|\,d\omega .$$ [5].

$\boxed{B}$ For every $-T \le t \le T$,

$$\left| f(t) - \sum_{n=-N}^{N} f(nT)\, \operatorname{sinc}\left(\frac{t-nT}{T}\right)\right| \le 2 \int_{|\omega|>\frac{1}{2T}} |F(\omega)|\,d\omega + \frac{1}{\pi}\sqrt{\frac{2}{N-1}} \left(\sum_{|n|>N} |f(nT)|^2\right)^{1/2}$$

$\boxed{C}$ Choose M_0 an integer. For every $(M_0-1)T \le t \le (M_0+1)T$,

$$\left| f(t) - \sum_{n=M_0-N}^{M_0+N} f(nT)\, \operatorname{sinc}\left(\frac{t-nT}{T}\right)\right| \le 2 \int_{|\omega|>\frac{1}{2T}} |F(\omega)|\,d\omega + \frac{1}{\pi}\sqrt{\frac{2}{N-1}} \left(\sum_{n<M_0-N} |f(nT)|^2 + \dots\right.$$

$$\left. \dots + \sum_{n>M_0+N} |f(nT)|^2 \right)^{1/2}.$$

(B is the special case of C, when $M_0 = 0$).

$\boxed{D}$ For every $-T \le t \le T$, and $\omega_0 \in \mathbb{R}$

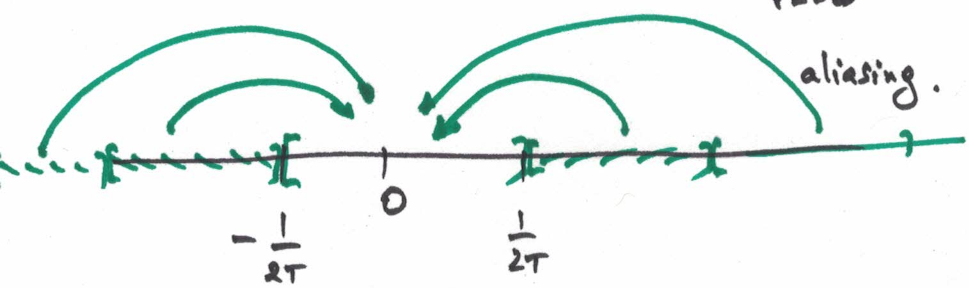
$$\left| f(t) - \sum_{n=-N}^{N} f(nT)\, e^{2\pi i \omega_0 (t-nT)} \operatorname{sinc}\left(\frac{t-nT}{T}\right)\right| \le 2 \int_{|\omega-\omega_0|>\frac{1}{2T}} |F(\omega)|\,d\omega + \frac{1}{\pi}\sqrt{\frac{2}{N-1}} \left(\sum_{|n|>N} |f(nT)|^2\right)^{1/2}$$

Why: Recall Poisson Summation Formula.

$$\sum_{m=-\infty}^{\infty} f(x-mp) = \frac{1}{p}\sum_{k=-\infty}^{\infty} F\left(\frac{k}{p}\right) e^{2\pi i \frac{kx}{p}}$$

Take. $p=\frac{1}{T}$, switch $f \leftrightarrow F$, $k \to -k$.

$$\sum_{m=-\infty}^{\infty} F\left(\omega-\frac{m}{T}\right) = T\sum_{k=-\infty}^{\infty} f(kT)\, e^{-2\pi i k\omega T}, \quad \forall \omega.$$



Aliased $f(x)$ $= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} e^{2\pi i w x} \cdot \sum_m F\left(w-\frac{m}{T}\right) dw = \cdots = T \sum_k f(kT) \cdot \underbrace{\int_{-\frac{1}{2T}}^{\frac{1}{2T}} e^{2\pi i w (x-kT)} dw}_{\to \operatorname{sinc}\left(\frac{x-kT}{T}\right)}$.

Examples.

① Assume $f: \mathbb{R} \to \mathbb{R}$ is a 1 kHz - band limited signal.

We sample f with sampling period $T = 0.1\,ms = 100\,\mu s$.

Assume. $|f(t)| \leq \frac{1}{1+|t|}$.

How many terms in the Shannon's formula we need to utilize to achieve an accuracy of $\varepsilon = 10^{-2}$ for all $-0.1\,ms \leq t \leq 0.1\,ms$.

Solution:

Use part B : with $T = 0.1\,ms$, $\frac{1}{2T} = \frac{1}{0.2 \cdot 10^{-3}}\,Hz = 5000\,Hz$.

$$\left| f(t) - \sum_{n=-N}^{N} f(nT)\,\mathrm{sinc}\left(\frac{t-nT}{T}\right) \right| \leq 2 \underbrace{\int_{|\omega|>5000\,Hz} |F(\omega)|\,d\omega} + \frac{1}{\pi}\sqrt{\frac{2}{N-1}} \left(\sum_{|n|>N} |f(nT)|^2 \right)^{1/2}$$

$$\underbrace{\int_{-\infty}^{-5000} |F(\omega)|\,d\omega}_{=0} + \underbrace{\int_{5000}^{\infty} |F(\omega)|\,d\omega}_{=0} = 0.$$

$$|\text{error}| \leq \frac{1}{\pi}\sqrt{\frac{2}{N-1}} \left(2 \sum_{n=N+1}^{\infty} \frac{1}{(1+nT)^2} \right)^{1/2} = \underbrace{\frac{2}{\pi} \frac{1}{\sqrt{N-1}} \left(\sum_{n=N+1}^{\infty} \frac{1}{(1+nT)^2} \right)^{1/2}}_{\leq\, \varepsilon = 10^{-2}}$$

Want : $$\frac{1}{N-1} \sum_{n=N+1}^{\infty} \frac{1}{(1+n \cdot 10^{-4})^2} \leq \frac{\pi^2}{4} \cdot 10^{-4}$$

(8).

Approximate: $\sum_{n=N+1}^{\infty} \frac{1}{(1+n\cdot 10^{-4})^2} \approx \int_N^{\infty} \frac{dx}{(1+x\cdot 10^{-4})^2} = 10^4 \cdot \int_{N\cdot 10^{-4}}^{\infty} \frac{dy}{(1+y)^2} =$

$y = x\cdot 10^{-4}$

$$= 10^4 \cdot \frac{1}{1+N\cdot 10^{-4}} = \frac{10^8}{N+10^4}.$$

$$\frac{1}{N-1} \cdot \frac{10^8}{N+10^4} \le \frac{\pi^2}{4} \cdot 10^{-4}$$

$$N^2 + 10^4 \cdot N \ge \frac{4}{\pi^2} \frac{10^8}{10^{-4}} = \frac{4}{\pi^2} 10^{12}.$$

$$\ldots \rightarrow N \simeq 637{,}000 \text{ samples}.$$
