L15

Approximations by Cardinal Series

Example. Assume $f: \mathbb{R} \to \mathbb{R}$ is such that,

$$|F(\omega)| \le \frac{1}{(1+|\omega|)^2}, \quad \text{for every } \omega \in \mathbb{R}.$$

Suppose we sample f with a sampling period $T = 1\,ms$.
What is the maximum error if we approximate f using the entire cardinal series?

technically it should be sup (supremum).

Solution.

[Question]: Estimate

$$\max_{t \in \mathbb{R}} \left| f(t) - \sum_{n=-\infty}^{\infty} f(nT)\, \mathrm{sinc}\left(\frac{t-nT}{T}\right) \right|.$$

Using part A:

$$\left| f(t) - \sum_{n=-\infty}^{\infty} f(nT)\, \mathrm{sinc}\left(\frac{t-nT}{T}\right) \right| \le 2 \cdot \int_{|\omega| > \frac{1}{2T}} |F(\omega)|\, d\omega \le$$

$$\le 2\left[\int_{-\infty}^{-\frac{1}{2T}} |F(\omega)|\, d\omega + \int_{\frac{1}{2T}}^{\infty} |F(\omega)|\, d\omega \right] = 4 \cdot \int_{\frac{1}{2\cdot 10^{-3}}}^{\infty} |F(\omega)|\, d\omega \le$$

$$\le 4 \int_{500}^{\infty} \frac{1}{(1+\omega)^2}\, d\omega = 4\left(\frac{-1}{1+\omega}\right)\Bigg|_{500}^{\infty} = \frac{4}{501} \approx \frac{8}{1000} = 0.008$$

Generalized Functions. Distributions

Function:

Domain of Definition: "Set".

Domain of Values, Codomain: "Set"

Association: $x \in$ Domain $\longmapsto y \in$ Codomain.

Example: Fourier Transform.

$\mathcal{F}: L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$

$\mathcal{F}: f \longmapsto F$

$\mathcal{F}: L^1(\mathbb{R}) \longrightarrow L^\infty(\mathbb{R})$

(technically: $\mathcal{F}: L^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$.

where $C_0(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{C},\ f$ continuous and $\lim_{x \to \pm\infty} f(x) = 0 \}$

A generalized function (or distribution) is a map from a set of test functions to $\mathbb{C}$,

$$f: \text{Set of Test Functions} \longrightarrow \mathbb{C}$$

such that:

1) f is linear, $f(a_1\phi_1 + a_2\phi_2) = a_1 f(\phi_1) + a_2 f(\phi_2)$

where $a_1, a_2 \in \mathbb{C}$

ϕ_1, ϕ_2 are test functions.

2) f is "continuous".

Test Functions.

For our class we focus on. the class of Schwartz functio[ns]

A function $\Phi : \mathbb{R} \longrightarrow \mathbb{C}$ is called a Schwartz function if:

1) Φ is $C^\infty(\mathbb{R})$, i.e. infinitely many time differentiable, with continuous derivative each time.

2) $\phi, \phi', \phi'', \ldots$ decays faster than any polynomial.

②: For every $N \geq 0$, for every $M \geq 0$, there is $C_{N,M,\phi} \geq 0$ such that:

$$(1+|x|)^M \cdot \left| \Phi^{(N)}(x) \right| \leq C_{N,M,\phi} \text{, for every } x \in \mathbb{R}$$

Notation: $\mathcal{S}$ denotes the set of Schwartz functions.

Example. $\phi(x) = e^{-x^2}$ is a test function.

Why:

1) Is $\phi \in C^\infty$? Yes.

$$\phi'(x) = -2x\, e^{-x^2}, \quad \phi''(x) = -2e^{-x^2} + 4x^2 e^{-x^2},$$

$$\phi'''(x) = 4x\, e^{-x^2} + 8x\, e^{-x^2} - 8x^3 e^{-x^2} = (12x - 8x^3) e^{-x^2}$$

$\ldots$

$$\phi^{(N)}(x) = P_N(x) \cdot e^{-x^2}, \text{ where } P_N \text{ is a polynomial of degree } N.$$

2).

$$(1+|x|)^M \cdot |\phi^{(N)}(x)| = (1+|x|)^M \cdot \underbrace{P_N(x)}_{\text{poly of deg. } N} \cdot e^{-x^2} =$$

$$= \underbrace{\frac{(1+|x|)^M P_N(x)}{1+|x|^{M+N}}}_{\text{bounded}} \cdot \underbrace{(1+|x|^{M+N}) \cdot e^{-x^2}}$$

bounded: $\left| \frac{(1+|x|)^M P_N(x)}{1+|x|^{M+N}} \right| \leq C_1$, for every x.

C_1 depends on M and N

$$\lim_{x\to\pm\infty} (1+|x|^{M+N}) \cdot e^{-x^2} = ?$$

$$\lim_{x\to\infty} (1+x^{M+N}) e^{-x^2} = \lim_{x\to\infty} \frac{x^{M+N}}{e^{x^2}} =$$

by l'Hospital

$$\overset{\downarrow}{=} \lim_{x\to\infty} \frac{(M+N)x^{M+N-1}}{2x \cdot e^{x^2}} = \cdots = \lim_{x\to\infty} \frac{(M+N)!}{\underbrace{(e^{x^2})^{(M+N)}}_{Q_{M+N}(x) \cdot e^{x^2}}} = 0.$$

$$\longrightarrow \left| (1+|x|^{M+N}) \cdot e^{-x^2} \right| \leq C_2,$$

for every x

$$\Rightarrow (1+|x|)^M \cdot |\phi^{(N)}(x)| \leq C_1 \cdot C_2 = C(M,N,\phi).,$$ for every $x \in \mathbb{R}$

Examples:

For any polynomial P, e.g. $P(x) = x^2 + 2x + 22$.

any real numbers $a > 0$, $b \in \mathbb{R}$.

$$\phi(x) = P(x) \cdot e^{-ax^2 + bx}$$

$\rightarrow$ Such ϕ is a test function.

$$a = 2,\ b = -7 \Rightarrow \phi(x) = (x^2 + 2x + 22) \cdot e^{-2x^2 - 7x}.$$

Property. If ϕ_1 and ϕ_2 are test functions, and $a_1, a_2 \in \mathbb{C}$ then $a_1 \phi_1 + a_2 \cdot \phi_2$ is also a test function.

$\Rightarrow \mathcal{S}$ is a $\mathbb{C}$-vector space.

Example:

Is $\phi(x) = x$ a test function?

a) ϕ is C^∞

b). ϕ is not bounded.

$\not\rightarrow$ ϕ is not a test function.

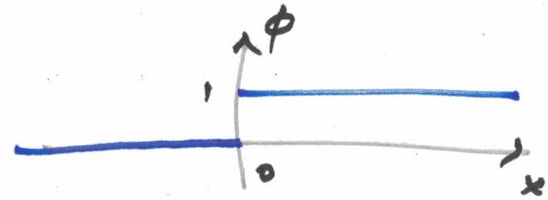
Example. $\phi(x) = x^2 + 2x + 22$, or any polynomial (other than $P = 0$)

$\hookrightarrow$ such ϕ is not a test function.

$\phi : R \to R$

Example:

$$\phi(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

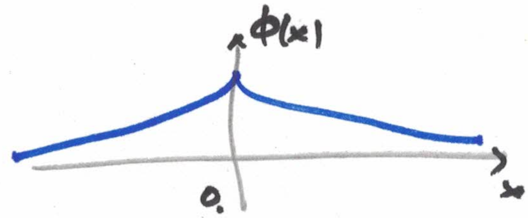


$\downarrow$

NOT a test function.

Example

$$\phi(x) = e^{-|x|}.$$



$$\lim_{M \to \infty} \left(x^M \cdot \phi(x) \right) = \lim_{x \to \infty} \frac{x^M}{e^x} = \cdots = \lim_{x \to \infty} \frac{M!}{e^x} = 0.$$

This is continuous but not **differentiable** on R.

$\hookrightarrow$ not differentiable at 0.

ϕ is **NOT** a test function.

Example.

$$\phi(x) = \sin(x) \quad \left(\text{or} \quad \cos(7x). \right).$$

$\downarrow$

ϕ is **not** a test function.

Property: If ϕ_1, ϕ_2 are test functions then $\phi_1 \cdot \phi_2$ is also a test function.

Why:

$$(\phi_1 \cdot \phi_2)' = \phi_1' \cdot \phi_2 + \phi_1 \cdot \phi_2'$$

$$(\phi_1 \cdot \phi_2)'' = \phi_1'' \cdot \phi_2 + 2\phi_1' \cdot \phi_2' + \phi_1 \cdot \phi_2''$$

(7)

$$(\phi_1 \cdot \phi_2)''' = \phi_1''' \cdot \phi_2 + 3\,\phi_1'' \cdot \phi_2' + 3\,\phi_1' \phi_2'' + \phi_1 \cdot \phi_2'''$$

Leibniz rule:

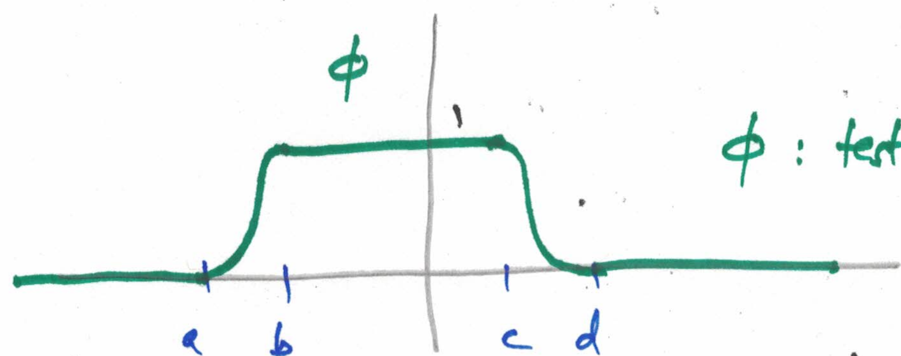
$$(\phi_1 \cdot \phi_2)^{(N)} = \binom{N}{0} \phi_1^{(N)} \cdot \phi_2 + \binom{N}{1} \phi_1^{(N-1)} \cdot \phi_2' + \dots + \binom{N}{N-1} \phi_1' \phi_2^{(N-1)} + \phi_1 \phi_2^{(N)}$$

$$= \sum_{k=0}^{N} \binom{N}{k} \phi_1^{(N-k)} \cdot \phi_2^{(k)}.$$

$\overline{\mathcal{J}}$ is closed to addition (vector space) and multiplication. (algebra).

Example.

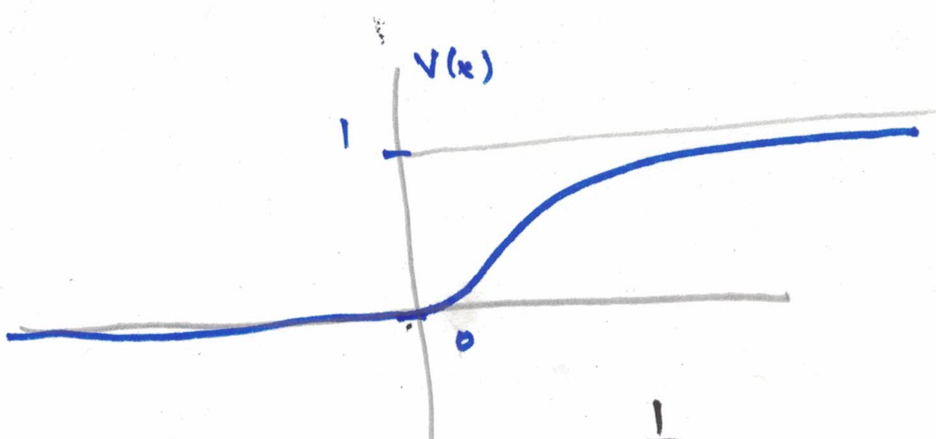
Wait:



ϕ : test function.

Step 1.

$$v : \mathbb{R} \longrightarrow \mathbb{R}, \qquad v(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \le 0. \end{cases}$$



Claim: $\underline{v \in C^\infty}$

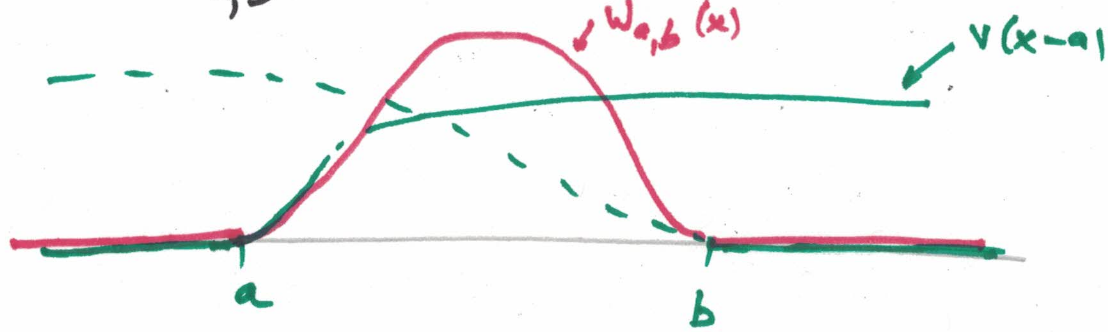
(but v is NOT a test function)

For $x>0$, $v'(x) = \frac{1}{x^2} \cdot e^{-\frac{1}{x}}$ $\dashrightarrow$ $\lim_{x \downarrow 0} v'(x) = 0$. (check).

$$v''(x) = -\frac{2}{x^3} e^{-\frac{1}{x}} + \frac{1}{x^4} e^{-1/x} = P\left(\frac{1}{x}\right) \cdot e^{-\frac{1}{x}}.$$

Step 2.

$$W_{a,b}(x) = V(x-a)\cdot V(b-x), \quad a<b$$



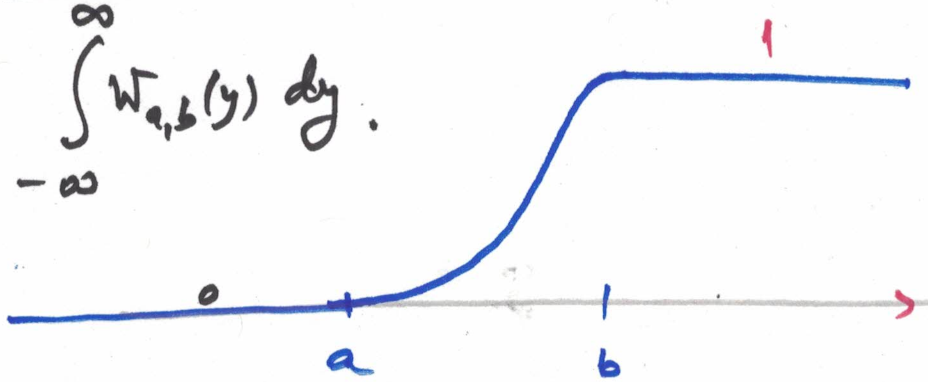
$$W_{a,b} \in C^{\infty}.$$

$\not\longrightarrow$ W is a test function.

$$\sup_x |x|^M \cdot |W^{(N)}(x)| < \infty.$$

Step 3.

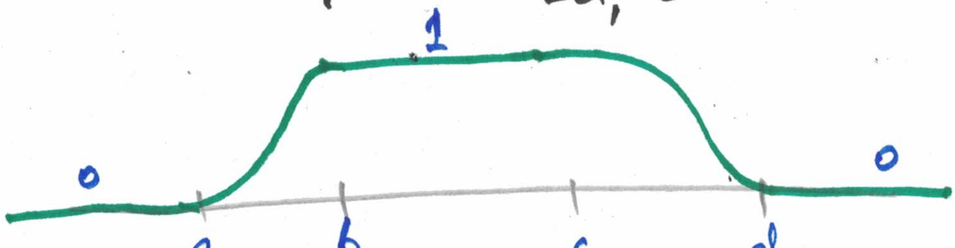
$$g_{a,b}(x) = \frac{\int_{-\infty}^{x} W_{a,b}(y)\,dy}{\int_{-\infty}^{\infty} W_{a,b}(y)\,dy}.$$



step 4.

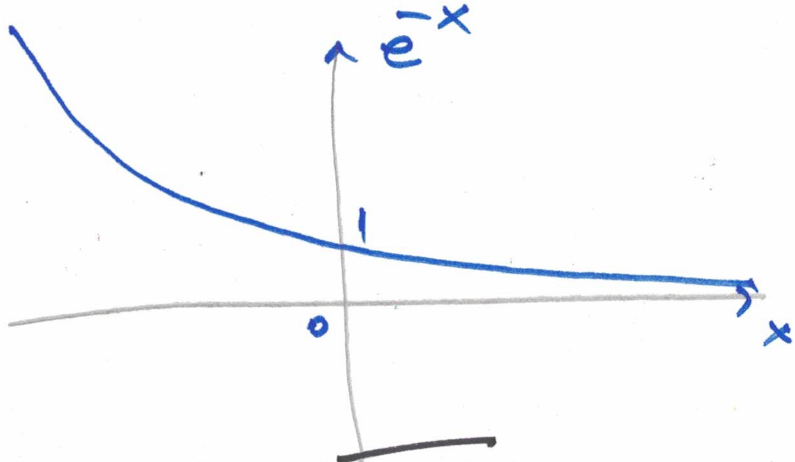
$$\phi(x) = g_{a,b}(x)\cdot g_{-d,-c}(x).$$ $\to$ ϕ is a test function

MESA function.



Example

$f(x) = e^{-x}$



f is C^{∞}.

$\sup_x |f(x)| = +\infty \longrightarrow \boxed{f \text{ is NOT}}$ a test function.

$C_{0,0}: \quad \underbrace{(1+|x|)^0}_{1} \cdot \underbrace{|f^{(0)}(x)|} \leq C_{0,0}.$

$f(x) \leq C_{0,0}$

$e^{-x} \leq C_{0,0}$, for every x.

$\lim_{x \to -\infty} e^{-x} = +\infty.$

$\left(e^{-ax^{2N} + c_1 x^{2N-1} + \dots c_{2N}} \right). \leftarrow$ is a test function.

$(a > 0)$

$e^{-\pi x^3} \rightarrow \lim_{x \to -\infty} e^{-x^3} = \infty$

$e^{-|x|^{1.5}}$.

$e^{-|x|^{1.5}} \leftarrow$ NOT a test function.