L18



Distribution $\frac{1}{x}$

Recall: $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \begin{cases} x \log(|x|) - x, & x \neq 0 \\ 0, & x = 0. \end{cases}$

CSG.

'$g'(x) = \log(|x|)$, in the sense of distributions:

$$g'\{\phi\} = \int_{-\infty}^{\infty} \log(|x|)\,\phi(x)\,dx = \underbrace{\int_{-\infty}^{0} \log(|x|)\,\phi(x)\,dx + \int_{0}^{\infty} \log(|x|)\,\phi(x)\,dx}$$

each integral converges as improper Riemann integrals.

$$\int_{-\infty}^{0} \log(|x|)\,\phi(x)\,dx = \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{-\varepsilon} \log(|x|)\,\phi(x)\,dx < \infty.$$

$$\int_{0}^{\infty} \log(|x|)\,\phi(x)\,dx = \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{\infty} \log(|x|)\,\phi(x)\,dx < \infty.$$

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$g'' = ?$ in the sense of distributions:

$$g''\{\phi\} = -g'\{\phi'\} = -\int_{-\infty}^{\infty} \log(|x|)\,\phi'(x)\,dx =$$

$$= -\lim_{\varepsilon \downarrow 0} \int_{-\infty}^{-\varepsilon} \log(|x|)\,\phi'(x)\,dx - \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{\infty} \log(|x|)\,\phi'(x)\,dx =$$

Integration by parts

$$= -\lim_{\varepsilon \downarrow 0}\left[\log(|x|)\phi(x)\Big|_{-\infty}^{-\varepsilon} - \int_{-\infty}^{-\varepsilon}\frac{1}{x}\phi(x)\,dx\right] -$$

For $x<0$

$$\frac{d}{dx}\log(|x|) \overset{\text{For } x<0}{=} \frac{d}{dx}\left(\log(-x)\right) = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$$

$$-\lim_{\varepsilon \downarrow 0}\left[\log(|x|)\phi(x)\Big|_{\varepsilon}^{\infty} - \int_{\varepsilon}^{\infty}\frac{1}{x}\phi(x)\,dx\right] \overset{*}{=}$$

Note: $\lim_{x\to\pm\infty}\left(\log(|x|)\phi(x)\right) = \lim_{x\to\pm\infty}\left(\underbrace{\frac{\log(|x|)}{x}}_{\downarrow\, 0}\cdot\underbrace{x\phi(x)}_{\downarrow\, 0}\right) = 0.$

$$\overset{*}{=} -\lim_{\varepsilon\downarrow 0}\left[\log(\varepsilon)\phi(-\varepsilon) - \int_{-\infty}^{-\varepsilon}\frac{\phi(x)}{x}\,dx\right] - \lim_{\varepsilon\downarrow 0}\left[-\log(\varepsilon)\phi(\varepsilon) - \int_{\varepsilon}^{\infty}\frac{\phi(x)}{x}\,dx\right] =$$

choose same $\varepsilon$

$$\overset{\downarrow}{=} -\lim_{\varepsilon\downarrow 0}\left[\underbrace{\log(\varepsilon)\phi(-\varepsilon) - \log(\varepsilon)\phi(\varepsilon)} - \int_{-\infty}^{-\varepsilon}\frac{\phi(x)}{x}\,dx - \int_{\varepsilon}^{\infty}\frac{\phi(x)}{x}\,dx\right] =$$

$$\lim_{\varepsilon\downarrow 0}\left[\log(\varepsilon)\cdot\left(\phi(\varepsilon)-\phi(-\varepsilon)\right)\right] = \lim_{\varepsilon\downarrow 0}\left[\underbrace{\left(2\varepsilon\,\log(\varepsilon)\right)}_{\downarrow\, 0}\cdot\underbrace{\left(\frac{\phi(\varepsilon)-\phi(-\varepsilon)}{2\varepsilon}\right)}_{\downarrow\, \varepsilon\downarrow 0.\ (\text{l'Hospital})\ \phi'(0)}\right] = 0.$$

$$= \lim_{\varepsilon\downarrow 0}\left[\int_{-\infty}^{-\varepsilon}\frac{\phi(x)}{x}\,dx + \int_{\varepsilon}^{\infty}\frac{\phi(x)}{x}\,dx\right].$$

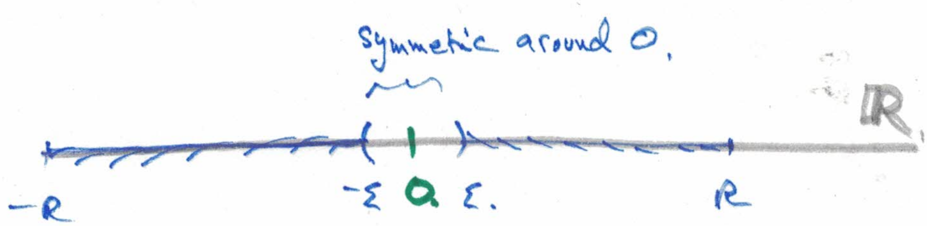
**Definition.** The principal value (p.v.) of an integral:

If $f : \mathbb{R} \to \mathbb{C}$ is a function that has a finite number of singularities, say at $x_1, x_2, \dots x_N$, $x_1 < x_2 < \dots < x_N$,

$$p.v. \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \lim_{\varepsilon_1 \downarrow 0} \lim_{\varepsilon_2 \downarrow 0} \dots \lim_{\varepsilon_N \downarrow 0} \left[ \int_{-R}^{x_1-\varepsilon_1} f(x)\,dx + \int_{x_1+\varepsilon_1}^{x_2-\varepsilon_2} f(x)\,dx + \int_{x_2+\varepsilon_2}^{x_3-\varepsilon_3} f(x)\,dx + \dots + \int_{x_{N-1}+\varepsilon_{N-1}}^{x_N-\varepsilon_N} f(x)\,dx + \int_{x_N+\varepsilon_N}^{R} f(x)\,dx \right]$$

For. $f(x) = \dfrac{\phi(x)}{x}$ ---> unique singularity at $x_1 = 0$:

$$p.v. \int_{-\infty}^{\infty} \frac{\phi(x)}{x}\,dx = \lim_{R\to\infty} \lim_{\varepsilon \downarrow 0} \left[ \int_{-R}^{-\varepsilon} \frac{\phi(x)}{x}\,dx + \int_{\varepsilon}^{R} \frac{\phi(x)}{x}\,dx \right]$$




**Definition**

$P_{-1} = \text{"} \quad , \quad P_{-1}\{\phi\} = p.v. \displaystyle\int_{-\infty}^{\infty} \frac{\phi(x)}{x}\,dx = \lim_{\varepsilon \downarrow 0} \left[ \int_{-\infty}^{-\varepsilon} \frac{\phi(x)}{x}\,dx + \int_{\varepsilon}^{\infty} \frac{\phi(x)}{x}\,dx \right.$

$=$

Equivalent formulae:

$$p.v. \int_{-\infty}^{\infty} \frac{\phi(x)}{x}\,dx = \lim_{R\to\infty.} \lim_{\varepsilon \downarrow 0} \left[ \int_{-R}^{-\varepsilon} \frac{\phi(x)}{x}\,dx + \int_{\varepsilon}^{R} \frac{\phi(x)}{x}\,dx \right]. \overset{*}{=}$$

$$\int_{-R}^{-\varepsilon} \frac{1}{x}\,dx + \int_{\varepsilon}^{R} \frac{1}{x}\,dx = \log(|x|)\Big|_{-R}^{-\varepsilon} + \log(x)\Big|_{\varepsilon}^{R} = \log(\varepsilon) - \log(R) + \log(R) -$$

$$-\log(\varepsilon) = 0.$$

$$\overset{*}{=} \lim_{R\to\infty} \lim_{\varepsilon \downarrow 0} \left[ \int_{-R}^{-\varepsilon} \frac{\phi(x) - \phi(0)}{x}\,dx + \int_{\varepsilon}^{R} \frac{\phi(x) - \phi(0)}{x}\,dx \right]. =$$

Note: $\lim_{x\to 0} \frac{\phi(x) - \phi(0)}{x} = \phi'(0).$

$$= \lim_{R\to\infty} \left[ \int_{-R}^{0} \frac{\phi(x) - \phi(0)}{x}\,dx + \int_{0}^{R} \frac{\phi(x) - \phi(0)}{x}\,dx \right] =$$

$$= \lim_{R\to\infty} \int_{-R}^{R} \frac{\phi(x) - \phi(0)}{x}\,dx = p.v. \int_{-\infty}^{\infty} \frac{\phi(x) - \phi(0)}{x}\,dx$$

(5).

$$\lim_{\varepsilon\downarrow 0}\left[\int_{-\infty}^{-\varepsilon}\frac{\phi(x)}{x}dx+\int_{\varepsilon}^{\infty}\frac{\phi(x)}{x}dx\right]=\lim_{\varepsilon\downarrow 0}\left[\int_{\infty}^{\varepsilon}\frac{\phi(-y)}{-y}(-dy)+\int_{-\varepsilon}^{-\infty}\frac{\phi(-y)}{-y}(-dy)\right]=$$

$x\to -x\ ,\quad y=-x.$

$$=-\lim_{\varepsilon\downarrow 0}\left[\int_{\varepsilon}^{\infty}\frac{\phi(-y)}{y}dy+\int_{-\infty}^{-\varepsilon}\frac{\phi(-y)}{y}dy\right]=-\lim_{\varepsilon\downarrow 0}\left[\int_{-\infty}^{-\varepsilon}\frac{\phi(-x)}{x}dx+\int_{\varepsilon}^{\infty}\frac{\phi(-x)}{x}dx\right]$$

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$$P_{-1}\{\phi\}=\frac{1}{2}\lim_{\varepsilon\downarrow 0}\left[\int_{-\infty}^{-\varepsilon}\frac{\phi(x)}{x}dx+\int_{\varepsilon}^{\infty}\frac{\phi(x)}{x}dx\right]-\frac{1}{2}\lim_{\varepsilon\downarrow 0}\left[\int_{-\infty}^{-\varepsilon}\frac{\phi(-x)}{x}dx+\int_{\varepsilon}^{\infty}\frac{\phi(-x)}{x}dx\right]$$

$$=\lim_{\varepsilon\downarrow 0}\left[\int_{-\infty}^{-\varepsilon}\frac{\phi(x)-\phi(-x)}{2x}dx+\int_{\varepsilon}^{\infty}\frac{\phi(x)-\phi(-x)}{2x}dx\right]=$$

$x=0$ is no longer a singularity (it is an apparent singularity),

$$=\int_{-\infty}^{\infty}\frac{\phi(x)-\phi(-x)}{2x}dx$$

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To summarize: $P_{-1}=g''$, is the distribution associated to $\frac{1}{x}$:

$$P_{-1}\{\phi\}=\text{p.v.}\int_{-\infty}^{\infty}\frac{\phi(x)}{x}dx=\text{p.v.}\int_{-\infty}^{\infty}\frac{\phi(x)-\phi(0)}{x}dx=\int_{-\infty}^{\infty}\frac{\phi(x)-\phi(-x)}{2x}dx$$

Assume $f(x) = x \cdot e^{-x^2}$

Question: ~~[redacted]~~ Is there a $\int_{-\infty}^{\infty} \frac{f(x)}{x}\,dx$ ~~have a singularity~~? singularity in this integral?

$$\frac{f(x)}{x} = e^{-x^2} \dashrightarrow \int_{-\infty}^{\infty} e^{-x^2}\,dx$$ → No singularity, here

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How to construct $\frac{1}{x^2}$ ? ... $\frac{1}{x^n}$ ?

Note: $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$ , for any $x \neq 0$. $\mapsto \frac{1}{x^2} = -\left(\frac{1}{x}\right)'$

(ordinary derivative).

Definition:

$$P_{-2} = -(P_{-1})' = -(P_{-1})'$$

...

[In an ordinary sense: $\left(\frac{1}{x^{n-1}}\right)' = (x^{-n+1})' = (-n+1)\cdot x^{-n} = -(n-1)\frac{1}{x^n}$

$$\frac{1}{x^n} = -\frac{1}{n-1}\cdot\left(\frac{1}{x^{n-1}}\right)'$$ ].

Definition: $P_{-n} = -\frac{1}{n-1}\left(P_{-(n-1)}\right)' = (-1)^{n-1}\frac{1}{(n-1)(n-2)\cdots 1}\cdot(P_{-1})^{(n-1)}$

We obtained: $\left[\; P_{-n} = \frac{(-1)^{n-1}}{(n-1)!}(P_{-1})^{(n-1)}\right.$ ← $(n-1)^{st}$ derivative of $P_{-1}$

What is

$$P_{-2}\{\phi\} = ?$$

$$P_{-2}\{\phi\} = P_{-1}\{\phi'\} = \text{p.v.} \int_{-\infty}^{\infty} \frac{\phi'(x) - \phi'(0)}{x}\, dx =$$

$$= \lim_{R\to\infty} \lim_{\varepsilon\to 0} \left[ \int_{-R}^{-\varepsilon} \frac{1}{x}\cdot \phi'(x)\, dx - \int_{-R}^{-\varepsilon} \frac{\phi'(0)}{x}\, dx + \int_{\varepsilon}^{R} \frac{1}{x}\phi'(x)\, dx - \int_{\varepsilon}^{R} \frac{\phi'(0)}{x}\, dx \right]$$

Int. by parts

$$\stackrel{\downarrow}{=} \lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \left[ \frac{\phi(x)}{x}\Big|_{-R}^{-\varepsilon} - \int_{-R}^{-\varepsilon} \frac{-1}{x^2}\phi(x)\, dx + \frac{\phi(x)}{x}\Big|_{\varepsilon}^{R} - \int_{\varepsilon}^{R} \frac{-1}{x^2}\phi(x)\, dx - \int_{-R}^{-\varepsilon} \frac{\phi'(0)}{x}\, dx - \int_{\varepsilon}^{R} \frac{\phi'(0)}{x}\, dx \right] =$$

$$\lim_{\varepsilon\to 0} \left( \frac{\phi(-\varepsilon)}{-\varepsilon} - \frac{\phi(\varepsilon)}{\varepsilon} \right) = -\lim_{\varepsilon\downarrow 0} \frac{\phi(\varepsilon) + \phi(-\varepsilon)}{\varepsilon} = \pm\infty.$$

$\Rightarrow$ IS NOT FINITE

$$= \lim_{R\to\infty} \left( -\frac{\phi(-R)}{-R} + \frac{\phi(R)}{R} \right) = 0.$$

$$= \lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \left[ \int_{-R}^{-\varepsilon} \frac{\phi(x)}{x^2}\, dx + \int_{\varepsilon}^{R} \frac{\phi(x)}{x^2}\, dx - \frac{\phi(-\varepsilon)}{\varepsilon} - \frac{\phi(\varepsilon)}{\varepsilon} - \int_{-R}^{-\varepsilon} \frac{\phi'(0)}{x}\, dx - \int_{\varepsilon}^{R} \frac{\phi'(0)}{x}\, dx \right.$$

$$\lim_{R\to\infty}\left[\int_{-R}^{-\varepsilon}\frac{\phi(0)}{x^2}\,dx+\int_{\varepsilon}^{R}\frac{\phi(0)}{x^2}\,dx.\right]=\lim_{R\to\infty}\left[-\frac{1}{x}\Big|_{-R}^{-\varepsilon}-\left(\frac{1}{x}\right)\Big|_{\varepsilon}^{R.}\right]\phi(0).\therefore$$

$$=\lim_{R\to\infty}\left[\left(\frac{1}{\varepsilon}-\frac{1}{R}-\frac{1}{R}+\frac{1}{\varepsilon}\right)\phi(0).\right]=\underline{\underline{2\,\frac{\phi(0)}{\varepsilon}}}$$

$$\Rightarrow\; p_{-2}\{\phi\}=\lim_{R\to\infty}\lim_{\varepsilon\downarrow 0}\left[\int_{-R}^{-\varepsilon}\left(\frac{\phi(x)}{x^2}-\frac{\phi(0)}{x^2}-\frac{\phi'(0)}{x}\right)dx+\int_{\varepsilon}^{R}\left(\frac{\phi(x)}{x^2}-\frac{\phi(0)}{x^2}-\frac{\phi'(0)}{x}\right)\right]$$

$$=\lim_{R\to\infty}\lim_{\varepsilon\downarrow 0}\left[\int_{-R}^{-\varepsilon}\frac{\phi(x)-(\phi(0)+x\cdot\phi'(0))}{x^2}\,dx+\int_{\varepsilon}^{R}\frac{\phi(x)-(\phi(0)+x\cdot\phi'(0))}{x^2}\,dx\right]$$

$$=p.v.\int_{-\infty}^{\infty}\frac{\phi(x)-(\phi(0)+x\cdot\phi'(0))}{x^2}\,dx.$$

To summarize:

$$P_{-2}\{\phi\}=p.v.\int_{-\infty}^{\infty}\frac{\phi(x)-(\phi(0)+x\cdot\phi'(0))}{x^2}\,dx$$

...

Similarly + induction:

$$\boxed{P_{-n}\{\phi\}=p.v.\int_{-\infty}^{\infty}\frac{\phi(x)-\left(\phi(0)+x\cdot\phi'(0)+\frac{1}{2}x^2\phi''(0)+\cdots+\frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0)\right)}{x^{n}}}$$