L20 (1)

Examples of convolutions

Given f a distributions, compute. $f * \delta$.

Method 1 : Using definition.

Choose (Fix) $\phi \in \mathcal{S}$, a test function.

For $x \in \mathbb{R}$, let $\psi_x(y) = \phi(x+y)$.

Step 1: $\alpha(x) = \delta\{\psi_x\} = \psi_x(0) = \phi(x) \Rightarrow \alpha = \phi$

Step 2. $(f * \delta)\{\phi\} = f\{\alpha\} = f\{\phi\}$.

$\underline{f * \delta = f}$.

Method 2. "Pretend" that $(f * \delta)(x) = \underbrace{\int_{-\infty}^{\infty} f(x-y)\,\delta(y)\,dy}$

pretend it is perform in an ordinary sense, or that $y \mapsto f(x-y)$ is a test function.

$$\underline{(f * \delta)(x)} = \int_{-\infty}^{\infty} f(x-y)\cdot\delta(y)\,dy = f(x-y)\Big|_{y=0} = \underline{f(x)}.$$

$\Rightarrow \underline{f * \delta = f}$.

Example.

let. $f(x) = \delta'(x)$, $g(x) = \delta(x-2)$

Question: $f * g = ?$

Solution: Apply method 2:

$$(f*g)(x) = \int_{-\infty}^{\infty} \delta'(x-y) \cdot \delta(y-2)\, dy = \delta'(x-y)\Big|_{y=2} = \underline{\delta'(x-2)}.$$

Example. $f * \delta' = ?$

Solution:

$$(f*\delta')(x) = \int_{-\infty}^{\infty} f(x-y) \cdot \delta'(y)\, dy = -\int_{-\infty}^{\infty} \frac{d}{dy}\big(f(x-y)\big) \cdot \delta(y)\, dy =$$

$$= -\int_{-\infty}^{\infty} f'(x-y) \cdot \left[\frac{d}{dy}(x-y)\right] \cdot \delta(y)\, dy = \int_{-\infty}^{\infty} f'(x-y) \cdot \delta(y)\, dy = f'(x-y)\Big|_{y=0} = f'(x)$$

↑ by chain rule

We obtained: $f * \delta' = f'$

[If g is a distribution: $g'\{\phi\} = -g\{\phi'\}$.].

Remark:

If f and g are distributions such that $f*g$ is a distribution, then:

$$(f*g)' = f'*g = f*g'$$

Why?

$$(f*g)'(x) = \frac{d}{dx}\int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy = \int_{-\infty}^{\infty} \frac{d}{dx}\left(f(x-y)\right)g(y)\,dy =$$

$$= \int_{-\infty}^{\infty} f'(x-y)\cdot g(y)\,dy = f'*g.$$

Since convolution is commutative: $f*g = g*f$

$$\Rightarrow \quad (g*f)' = g'*f. \qquad \Rightarrow (f*g)' = f'*g = f*g'$$

The Fourier Transform of (Tempered) Distributions

Definition. Let f be a distribution. Its **Fourier transform** $\hat{f}$ is the distribution given by:

$$\forall \phi \in \mathcal{S}, \quad \hat{f}\{\phi\} = f\{\hat{\phi}\}$$

where $\hat{\phi}$ denotes the usual Fourier transform of test function ϕ.

Why? The definition is based on the Plancherel - Parseval realtion.

[If f, g are two functions (e.g., in $L^2(\mathbb{R})$):

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle = \int_{-\infty}^{\infty} \hat{f}(s)\cdot\overline{\hat{g}(s)}\, ds.$$

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\cdot\overline{g(x)}\, dx$$

Take g such that: $\overline{\hat{g}(s)} = \phi(s)$. $\Rightarrow \langle \hat{f}, \hat{g}\rangle = \int_{-\infty}^{\infty} \hat{f}(s)\phi(s)\, ds = \hat{f}\{\phi\}$.

Left-hand side: $\int_{-\infty}^{\infty} f(x)\cdot\overline{g(x)}\, dx = f\{\bar{g}\}$. $=$

Need to figure out $\bar{g}$ in terms of ϕ

$$\bar{g}(x) = \overline{g(x)} = \overline{\int_{-\infty}^{\infty} e^{2\pi i x s}\cdot\hat{g}(s)\, ds} = \int_{-\infty}^{\infty} e^{-2\pi i x s}\cdot\overline{\hat{g}(s)}\, ds$$

$$= \int_{-\infty}^{\infty} e^{-2\pi i x s}\cdot\phi(s)\, ds = \hat{\phi}(x).$$

$\langle f, g\rangle = f\{\bar{g}\} = f\{\hat{\phi}\}$, $\quad \langle \hat{f}, \hat{g}\rangle = \hat{f}\{\phi\}$.

$$\Rightarrow \hat{f}\{\phi\} = f\{\hat{\phi}\}.$$

Examples.

1). If f is a CSG function that admits an ordinary Fourier transform,

i.e., $\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} f(x)\,dx$, and $\hat{f}$ is a CSG

then the Fourier transform ~~associ~~ of f in the sense of distributions

is the distribution associated to $\hat{f}$ (i.e., of the ordinary Fourier transform)

E.G., $f(x) = e^{-\pi x^2} \longrightarrow \hat{f}(s) = e^{-\pi s^2}$

$$\hat{f}\{\phi\} = \int_{-\infty}^{\infty} e^{-\pi s^2} \cdot \phi(s)\,ds,$$

$$f\{\hat{\phi}\} = \int_{-\infty}^{\infty} e^{-\pi x^2}\, \hat{\phi}(x)\,dx.$$

$$\longmapsto \int_{-\infty}^{\infty} e^{-\pi x^2}\, \hat{\phi}(x)\,dx = \int_{-\infty}^{\infty} e^{-\pi s^2}\, \phi(s)\,ds$$

2) $\hat{\delta} = ?$

$$\hat{\delta}\{\phi\} = \delta\{\hat{\phi}\} = \hat{\phi}(0) = \int_{-\infty}^{\infty} e^{-2\pi i x s} \cdot \phi(x)\,dx \Big|_{s=0} = \int_{-\infty}^{\infty} \phi(x)\,dx =$$

$$= \int_{-\infty}^{\infty} 1(x) \cdot \phi(x)\,dx = \mathbf{1}\{\phi\},$$

where $\mathbf{1}$ is the fundamental functional associated to the constant function $1 : \mathbb{R} \to \mathbb{R}$, $1(x) = 1$, $\forall x$.

Conclusion: $\hat{\delta} = \mathbf{1}$.

3) $\widehat{\delta'} = ?$

$$\widehat{\delta'}\{\phi\} = \delta'\{\hat{\phi}\} = -\delta\{(\hat{\phi})'\} = -\left((\hat{\phi})'\right)\Big|_{s=0} =$$

$$\phi \longrightarrow \hat{\phi}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s x}\,\phi(x)\,dx.$$

$$= -\left(\frac{d}{ds}\int_{-\infty}^{\infty} e^{-2\pi i s x}\,\phi(x)\,dx\right)\Bigg|_{s=0} = -\left(\int_{-\infty}^{\infty}\frac{d}{ds}\left(e^{-2\pi i s x}\right)\phi(x)\,dx\right)\Bigg|_{s=0} =$$

$$= -\int_{-\infty}^{\infty}(-2\pi i x)\,e^{-2\pi i s x}\,\phi(x)\,dx\,\Bigg|_{s=0} = \int_{-\infty}^{\infty} 2\pi i x\cdot\phi(x)\,dx$$

$$\Rightarrow \boxed{\widehat{\delta'}(x) = 2\pi i x}$$

4). Rules for Fourier transform apply to distributions as well.

In particular: $\widehat{f*g} = \hat{f}\cdot\hat{g}$.

[Consistent with ~~the~~ $\hat{\delta} = 1$:

We have seen: $f*\delta = f$. $\longrightarrow$ $\widehat{f*\delta} = \hat{f}\cdot\hat{\delta} = \hat{f}$ $\Longleftrightarrow$ $\hat{\delta} = 1$.]

Method 2:

i) $$\hat{\delta}(s) = \underbrace{\int_{-\infty}^{\infty} e^{-2\pi i x s} \cdot \delta(x)\, dx}_{\text{Pretend it is an ordinary integral.}} = e^{-2\pi i x \cdot s}\Big|_{x=0} = 1, \text{ for all } s.$$

$$\Rightarrow \underline{\hat{\delta} = 1}.$$

ii). $$\hat{\delta'}(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} \cdot \delta'(x)\, dx = -\int_{-\infty}^{\infty} \frac{d}{dx}\left(e^{-2\pi i x s}\right) \cdot \delta(x)\, dx =$$

$$= -\int_{-\infty}^{\infty} (-2\pi i s)\, e^{-2\pi i x s} \cdot \delta(x) dx = 2\pi i s \underbrace{\int_{-\infty}^{\infty} e^{-2\pi i x s} \cdot \delta(x) dx}_{1.} = 2\pi i s$$

$$=$$

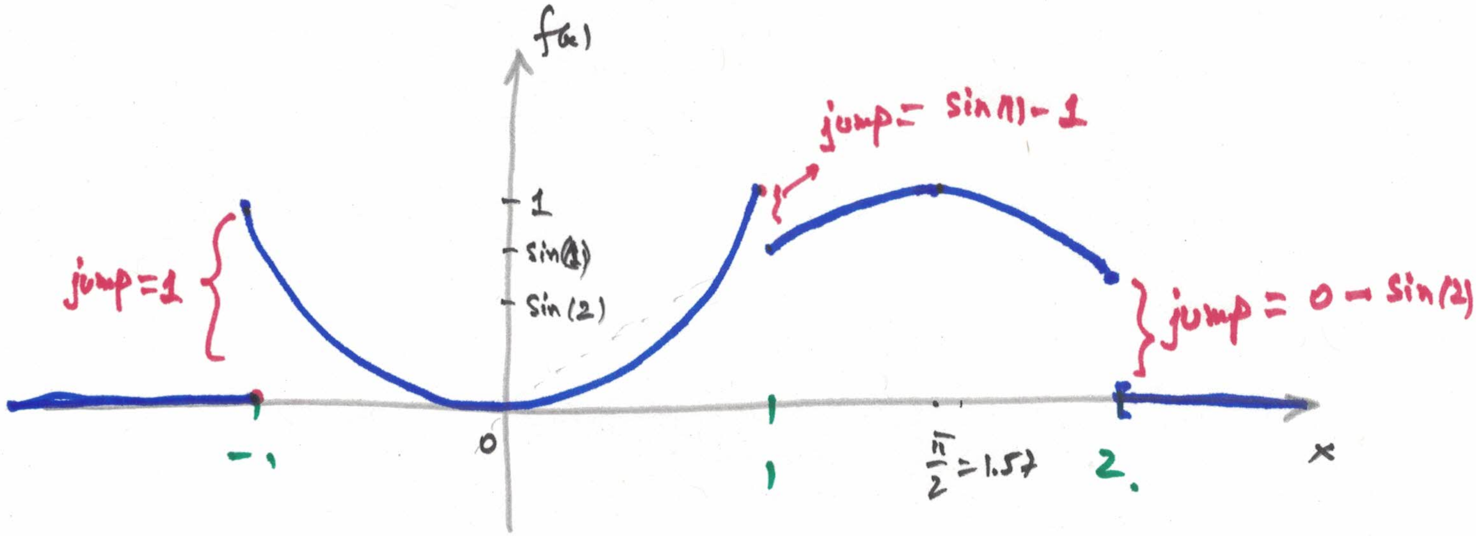
$$\underline{\hat{\delta'}(s) = 2\pi i s}$$

Derivatives of Distributions

Example:

Assume $f: \mathbb{R} \to \mathbb{R}$, $\quad f(x) = \begin{cases} 0, & x < -1. \\ x^2, & -1 \leq x \leq 1 \\ \sin(x), & 1 < x < 2 \\ 0, & x \geq 2. \end{cases}$

Want: f', f'', f''' in the sense of distributions.



$f' = ?$ Pick $\phi \in \mathcal{J}$ a test function.

$$f'\{\phi\} = -f\{\phi'\} = -\int_{-\infty}^{\infty} f(x)\cdot\phi'(x)\,dx = -\int_{-\infty}^{-1} f(x)\phi'(x)\,dx - \int_{-1}^{1} f(x)\phi'(x)\,dx -$$

$$-\int_{1}^{2} f(x)\phi'(x)\,dx - \int_{2}^{\infty} f(x)\phi'(x)\,dx = -\int_{-1}^{1} x^2\cdot\phi'(x)\,dx - \int_{1}^{2} \sin(x)\phi'(x)\,dx =$$

$$= -\left[x^2\cdot\phi(x)\Big|_{-1}^{1} - \int_{-1}^{1} 2x\cdot\phi(x)\,dx\right] - \left[\sin(x)\,\phi(x)\Big|_{1}^{2} - \int_{1}^{2} \cos(x)\,\phi(x)\,dx\right] =$$

$$= -\left[\phi(1) - \phi(-1)\right] + \int_{-1}^{1} 2x\,\phi(x)\,dx \;-\left[\sin(2)\,\phi(2) - \sin(1)\cdot\phi(1)\right] + \int_{1}^{2} \cos(x)\phi(x)dx$$

$$= \int_{-1}^{1} 2x\,\phi(x)\,dx + \int_{1}^{2} \cos(x)\,\phi(x)\,dx \;+\; \phi(-1) + \left(\sin(1) - 1\right)\phi(1) - \sin(2)\cdot\phi(2)$$

$$= \int_{-\infty}^{\infty} g_1(x)\cdot\phi(x)\,dx \;+\; \int_{-\infty}^{\infty}\left[\delta(x+1) + \left(\sin(1)-1\right)\cdot\delta(x-1) - \sin(2)\cdot\delta(x-2)\right]\phi(x)$$

where, $g_1 : \mathbb{R} \to \mathbb{R}$, $g_1(x) = \begin{cases} 0, & x < -1 \\ 2x, & -1 < x < 1 \\ \cos(x), & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

$$\Rightarrow \quad f'(x) = g_1(x) + (1-0)\cdot\delta(x+1) + (\sin(1)-1)\cdot\delta(x-1) + (0-\sin(2))\,\delta(x-2)$$

so that $f'\{\phi\} = \int_{-\infty}^{\infty}\left(g_1(x) + \;\ldots\;\right)\phi(x)\,dx$

In general: If $\quad f(x) = \begin{cases} f_1(x), & x < x_1 \\ f_2(x), & x_1 < x < x_2 \\ f_3(x), & x_2 < x < x_3 \\ \vdots & \\ f_N(x), & x_{N-1} < x < x_N \\ f_{N+1}(x), & x > x_N \end{cases}$

Then:

$$f'(x) = g(x) + g_{discrete}(x).$$

where

$$g(x) = \begin{cases} f_1'(x), & x < x_1 \\ f_2'(x), & x_1 < x < x_2 \\ \vdots & \\ f_N'(x), & x_{N-1} < x < x_N \\ f_{N+1}'(x), & x > x_N \end{cases}$$

$$g_{discrete}(x) = \left(f_2(x_1+0) - f_1(x_1-0)\right) \cdot \delta(x - x_1) + \left(f_3(x_2+0) - f_2(x_2-0)\right) \cdot$$

$$\cdot \delta(x - x_2) + \cdots + \left(f_{N+1}(x_N+0) - f_N(x_N-0)\right) \cdot \delta(x - x_N).$$

$$= \sum_{k=1}^{N} jump(x_k) \cdot \delta(x - x_k)$$

where $jump(x_k) = f(x_k+0) - f(x_k-0)$