L21

Last Time: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 0, & x < -1 \\ x^2, & -1 \le x \le 1 \\ \sin(x), & 1 < x < 2 \\ 0, & x \geq 2. \end{cases}$

$f'(x) = g(x) + g_{discrete}(x)$, $g(x) = \begin{cases} 0, & x < -1 \\ 2x, & -1 < x < 1 \\ \cos(x), & 1 < x < 2 \\ 0, & x > 2. \end{cases}$

$g_{discrete}(x) = \delta(x+1) + (\sin(1) - 1)\cdot\delta(x-1) + (-\sin(2))\cdot\delta(x-2)$

Now: compute f'', f''':

$f'' = g' + g'_{discrete} = \begin{cases} 0, & x < -1 \\ 2, & -1 < x < 1 \\ -\sin(x), & 1 < x < 2 \\ 0, & x > 2 \end{cases} + (-2)\cdot\delta(x+1) + (\cos(1) - 2)\cdot\delta(x-1) +$

$+ (-\cos(2))\cdot\delta(x-2) + \delta'(x+1) + (\sin(1) - 1)\cdot\delta'(x-1) + (-\sin(2))\cdot\delta'(x-2)$

$f''' = \begin{cases} 0, & x < 1 \\ -\cos(x), & 1 < x < 2 \\ 0, & x > 2. \end{cases} + 2\,\delta(x+1) + (-\sin(1) - 2)\,\delta(x-1) + \sin(2)\cdot\delta(x-2)$

$- 2\,\delta'(x+1) + (\cos(1) - 2)\,\delta'(x-1) - \cos(2)\,\delta'(x-2) + \delta''(x+1) + (\sin(1) - 1)\,\delta''(x-1) -$

$- \sin(2)\cdot\delta''(x-2)$.

More examples of Fourier Transform

1. $p_n(x) = x^n \cdot 1$

see rules for computing Fourier transform.

$$\hat{p}_n(s) \stackrel{\swarrow}{=} \frac{1}{(-2\pi i)^n} \cdot \frac{d^n}{ds^n}\left(\hat{1}\right)(s)$$

$$\hat{p}_n\{\phi\} = \int_{-\infty}^{\infty} \hat{p}_n(s) \cdot \phi(s)\, ds.$$

$$1 \longrightarrow \hat{1}\{\phi\} = 1\{\hat{\phi}\} = \int_{-\infty}^{\infty} \hat{\phi}(s) ds = \phi(0) = \delta\{\phi\}.$$

Hence: $\hat{1}(s) = \delta(s)$

$$\frac{d^n}{ds^n}\left(\hat{1}(s)\right) = \delta^{(n)}(s).$$

$$\Rightarrow \quad \underline{\hat{p}_n = \frac{1}{(-2\pi i)^n} \cdot \delta^{(n)}}$$

2. $f(x) = \sin(\omega_0 \cdot x) \longrightarrow \hat{f} = ?$

$\omega_0 \in \mathbb{R}$, $g(x) = \cos(\omega_0 \cdot x) \longrightarrow \hat{g} = ?$

$$f(x) = \frac{1}{2i}\left(e^{i\omega_0 x} - e^{-i\omega_0 x}\right), \qquad g(x) = \frac{1}{2}\left(e^{i\omega_0 x} + e^{-i\omega_0 x}\right)$$

Let $k(x) = e^{i\omega_0 x} \rightarrow \hat{k} = ?$

$$\hat{k}\{\phi\} = k\{\hat{\phi}\} = \int_{-\infty}^{\infty} e^{i\omega_0 x} \hat{\phi}(x)\, dx = \phi\left(\frac{\omega_0}{2\pi}\right) = \int_{-\infty}^{\infty} \delta\left(x - \frac{\omega_0}{2\pi}\right)\phi(x)dx$$

$$\Rightarrow \hat{k}(s) = \delta\left(s - \frac{\omega_0}{2\pi}\right).$$

Thus:

$$\hat{f}(s) = \frac{1}{2i}\left(\delta\left(s - \frac{\omega_0}{2\pi}\right) - \delta\left(s + \frac{\omega_0}{2\pi}\right)\right)$$

$$\hat{g}(s) = \frac{1}{2}\left(\delta\left(s - \frac{\omega_0}{2\pi}\right) + \delta\left(s + \frac{\omega_0}{2\pi}\right)\right).$$

3. $\hat{p}_{-1} = ?$, $p_{-1} \to$ distribution associated to $\frac{1}{x}$

$$p_{-1}\{\phi\} = \text{p.v.}\int_{-\infty}^{\infty} \frac{\phi(x)}{x}\,dx = \text{p.v.}\int_{-\infty}^{\infty} \frac{\phi(x)-\phi(0)}{x}\,dx = \text{p.v.}\int_{-\infty}^{\infty} \frac{\phi(x)-\phi(-x)}{2x}$$

$$\hat{p}_{-1}\{\phi\} = p_{-1}\{\hat{\phi}\} = \frac{1}{2}\,\text{p.v.}\int_{-\infty}^{\infty} \underbrace{\frac{\hat{\phi}(s)-\hat{\phi}(-s)}{s}}_{f(s):\ f(-s)=f(s)}\,ds. =$$

$$= \int_{0}^{\infty} \frac{\hat{\phi}(s)-\hat{\phi}(-s)}{s}\,ds =$$

$$= \int_{0}^{\infty} \frac{1}{s}\left[\left(\int_{-\infty}^{\infty} e^{-2\pi i s x}\phi(x)\,dx\right) - \left(\int_{-\infty}^{\infty} e^{2\pi i s x}\phi(x)\,dx\right)\right] ds =$$

$$= \int_{-\infty}^{\infty}\left(\int_{0}^{\infty} \frac{e^{-2\pi i s x} - e^{2\pi i s x}}{s}\,ds\right)\phi(x)\,dx = \int_{-\infty}^{\infty}\left(\int_{0}^{\infty} \underbrace{\frac{-2i\cdot\sin(2\pi s x)}{s}}\,ds\right)\phi(x)\,dx$$

$$g(s) = \frac{\sin(2\pi s x)}{s} : g(-s) = g(s)$$

$$= -i\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \frac{\sin(2\pi s x)}{s}\,ds\right)\phi(x)\,dx$$

Evaluate the inner integral:

$$\int_{-\infty}^{\infty} \frac{\sin(2\pi s x)}{s}\, ds = ?$$

i) ~~Assume~~ For $x > 0$:

$$\int_{-\infty}^{\infty} \frac{\sin(2\pi s x)}{s}\, ds = \int_{-\infty}^{\infty} \frac{\sin(\pi y)}{\cancel{\frac{1}{2x}} y} \cancel{\frac{1}{2x}}\, dy. = \pi \int_{-\infty}^{\infty} \frac{\sin(\pi y)}{\pi y}\, dy =$$

$$s \to y = 2s \cdot x, \quad s = \frac{1}{2x} \cdot y$$

$$= \pi \cdot \int_{-\infty}^{\infty} \operatorname{sinc}(y)\, dy = \pi \cdot \underbrace{\widehat{\operatorname{sinc}}(0)}_{\Pi(0) = 1} = \pi \quad (= 3.1415 \ldots)$$

ii) For $x < 0$:

$$\int_{-\infty}^{\infty} \frac{\sin(2\pi s x)}{s}\, ds = \int_{+\infty}^{-\infty} \frac{\sin(\pi y)}{\cancel{\frac{1}{2x}} y} \cancel{\frac{1}{2x}}\, dy. = -\int_{-\infty}^{\infty} \frac{\sin(\pi y)}{y}\, dy = -\pi$$

$$s \to y = 2sx, \quad s = \frac{1}{2x} y$$

iii) If $x = 0 \longrightarrow \int_{-\infty}^{\infty} \frac{\sin(2\pi s x)}{s}\, ds = 0.$

To summarize:

$$\int_{-\infty}^{\infty} \frac{\sin(2\pi s x)}{s}\, ds = \pi \cdot \operatorname{sign}(x) = \pi \cdot \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0. \end{cases}$$

Thus: $\hat{P}_{-1}\{\phi\} = -i \int_{-\infty}^{\infty} \pi \operatorname{sign}(x) \cdot \phi(x)\, dx \Rightarrow$

$$\Rightarrow \boxed{\hat{P}_{-1}(s) = -i\pi \operatorname{sign}(s)}$$

4). $$Ш(x) = \sum_{n=-\infty}^{\infty} \delta(x-n), \quad \hat{Ш} = ?$$

$$\hat{Ш}\{\phi\} = Ш\{\hat{\phi}\} = \sum_{n=-\infty}^{\infty} \hat{\phi}(n).$$

Recall the Poisson Summation Formula:

$$\sum_{n=-\infty}^{\infty} \phi(x-n) = \sum_{m=-\infty}^{\infty} \hat{\phi}(m)\cdot e^{2\pi i m x}$$

at $x=0$: $$\sum_{m=-\infty}^{\infty} \hat{\phi}(m) = \sum_{n=-\infty}^{\infty} \phi(0-n) = \sum_{n=-\infty}^{\infty} \phi(n)$$

$$\Rightarrow \quad \hat{Ш}\{\phi\} = \sum_{n=-\infty}^{\infty} \phi(n) = Ш\{\phi\}.$$

Summary: $$\hat{Ш} = Ш$$

Equivalently: $$\hat{Ш}(s) = \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} \delta(x-n)\right) e^{-2\pi i x s}\, dx = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x-n)\, e^{-2\pi i x s}\, dx =$$

$$= \sum_{n=-\infty}^{\infty} e^{-2\pi i n s} = \underbrace{\sum_{n=-\infty}^{\infty} e^{2\pi i n s}}_{\text{This is NOT a convergent series.}}$$

$$\Rightarrow \quad \boxed{\sum_{n=-\infty}^{\infty} e^{2\pi i n s} = \sum_{n=-\infty}^{\infty} \delta(s-n)}$$

or: $$\boxed{\sum_{n=-\infty}^{\infty} e^{2\pi i n x} = \sum_{n=-\infty}^{\infty} \delta(x-n)}$$

Use of Distributions in Solving Algebraic Equations

Typical Equation:

$$A(x)\cdot f(x) = g(x).$$

Given $A = A(x)$, a polynomial, $A(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n$

and g, a distribution, or, in some simpler cases, a CSG. function.

Want: f

1) One solution: $\frac{g}{A}$ $\dashrightarrow$ need to interpret as a distribution!

2) Are there other solutions?

Step 1. Factor $A(x) = a_0 \cdot \prod_{k=1}^{n} (x - z_k)$, $z_1, \dots, z_n \in \mathbb{C}$.

Factor using real zeros:

$$A(x) = B(x) \cdot \prod_{k=1}^{d} (x - x_k)^{m_k}$$

where. $x_1 \dots x_d \in \mathbb{R}$ are the distinct real zeros (roots) of A

$m_1 \dots m_k \geq 1$ are their multiplicities.

$B(x)$: polynomial with no real zeros.

Step 2

$$\frac{1}{A(x)} = \frac{1}{B(x)\cdot\prod_{k=1}^{d}(x-x_k)^{m_k}} = \frac{1}{B(x)} \sum_{k=1}^{d} \sum_{\ell=1}^{m_k} \frac{C_{k,\ell}}{(x-x_k)^{\ell}}$$

where $C_{k,\ell} \in \mathbb{C}$.

Example.:

$$\frac{1}{(x-1)^2\cdot(x-2)^3} = \frac{C_{1,1}}{x-1} + \frac{C_{1,2}}{(x-1)^2} + \frac{C_{2,1}}{(x-2)} + \frac{C_{2,2}}{(x-2)^2} + \frac{C_{2,3}}{(x-2)^3}$$

Step 3.

Define. $\frac{g}{A} \dashrightarrow \frac{g(x)}{B(x)} \sum_{k=1}^{d} \sum_{\ell=1}^{m_k} \frac{C_{k,\ell}}{(x-x_k)^{\ell}}$

as a distribution:

$$f_0(x) = \frac{g(x)}{B(x)} \sum_{k=1}^{d} \sum_{\ell=1}^{m_k} C_{k,\ell}\cdot P_{-\ell}(x-x_k).$$

where P_{-n} is the distribution associated to $\frac{1}{x^n}$.

specific solution.

Step 4.

General solution of the homogeneous equation. $A(x)\cdot f(x) = 0$.

$$f_h(x) = \sum_{k=1}^{d} \sum_{\ell=1}^{m_k} \lambda_{k,\ell}\cdot\delta^{(\ell-1)}(x-x_k) \quad , \lambda_{k,\ell} \in \mathbb{C}.$$

<u>Example</u>:

$$(x-1)\cdot f(x) = \sin(x)$$

A specific solution: $f_0(x) = \frac{\sin(x)}{x-1} = \sin(x)\cdot P_{-1}(x-1)$

The homogeneous equation: $(x-1)\cdot f(x) = 0.$

$$f_h(x) = \lambda\cdot\delta(x-1), \quad \lambda\in\mathbb{C}.$$

<u>why</u>:

$$g(x) = (x-1)\cdot\delta(x-1) = ?$$

$$g\{\phi\} = \int_{-\infty}^{\infty}(x-1)\,\delta(x-1)\,\phi(x)\,dx = \int_{-\infty}^{\infty}\delta(x-1)\,\big((x-1)\phi(x)\big)\,dx =$$

$$= \big((x-1)\cdot\phi(x)\big)\Big|_{x=1} = (1-1)\cdot\phi(1) = \underline{\underline{0}}. \quad ,\forall\phi\in\mathcal{S}.$$

$\Rightarrow g = 0$ (as a distribution). $\Rightarrow (x-1)\cdot\delta(x-1) = 0$

The general solution of: $(x-1)\cdot f(x) = \sin(x)$

$$\boxed{f(x) = f_0 + f_h = \sin(x)\cdot P_{-1}(x-1) + \lambda\cdot\delta(x-1)}$$

$\lambda\in\mathbb{C}.$

The general solution of : $A(x)\cdot f(x) = g(x)$

is the sum: $f(x) = \underbrace{f_0(x)}_{\text{specific solution}} + \underbrace{f_h(x)}_{\text{general solution of the homogeneous equation}}$.

with notations before,

$$f(x) = \frac{g(x)}{B(x)} \sum_{k=1}^{d} \sum_{\ell=1}^{m_k} c_{k,\ell}\, P_{-\ell}(x - x_k) + \sum_{k=1}^{d} \sum_{\ell=1}^{m_k} \lambda_{k,\ell} \cdot \delta^{(\ell-1)}_{(x-x_k)}$$

where $(c_{k,\ell})_{k,\ell}$ are uniquely determined by $A(x)$, $\lambda_{k,\ell} \in \mathbb{C}$.