Example Solve:

$$x^2 \cdot (x-2) \cdot f(x) = 0$$

in distributions.

Solution:

General Solution: $f(x) = a_1 \delta(x) + a_2 \cdot \delta'(x) + a_3 \cdot \delta(x-2)$

Why: let's check that each term, $\delta(x)$, $\delta'(x)$, $\delta(x-2)$ is a solution.

Need:

$\delta \longrightarrow x^2 \cdot (x-2) \cdot \delta(x) = 0$

$\delta(x-2) \longrightarrow x^2 \cdot (x-2) \cdot \delta(x-2) = 0.$

$\delta'(x) \longrightarrow x^2 (x-2) \delta'(x) = 0.$

Why

Take $\phi \in \mathcal{S}$,

$$\int_{-\infty}^{\infty} x^2 (x-2) \delta'(x) \phi(x) dx =$$

$$= -\int_{-\infty}^{\infty} \delta(x) \cdot \frac{d}{dx}\left[x^2 (x-2) \phi(x)\right] dx =$$

$$= -\int_{-\infty}^{\infty} \delta(x) \left[2x(x-2)\phi(x) + x^2 \phi(x) + x^2 (x-2) \phi'(x)\right] dx$$

$$= -[0 + 0 + 0] = 0.$$

Note: $P(x) \cdot \delta(x-x_0) = P(x_0) \cdot \delta(x-x_0)$

Why:

Take $\phi \in \mathcal{S}$,

$$\int_{-\infty}^{\infty} P(x) \delta(x-x_0) \phi(x) dx = \underline{P(x_0) \phi(x_0)}$$

$$\cancel{\int_{-\infty}^{\infty} P(x_0) \delta(x-x_0) \phi(x) dx = P(x_0) \phi(x_0)}$$

$$\int_{-\infty}^{\infty} P(x_0) \delta(x-x_0) \phi(x) dx = \underline{P(x_0) \phi(x_0)}$$

Solutions to Differential Equations

Example

Find the general solution of: $f'' - 4 . f = e^{-|x|}$
in the space of distributions

$f = f(x)$.

$$\frac{d^2 f}{dx^2}(x) - 4\, f(x) = e^{-|x|}$$

Solution.

Step 1. Let F denote the Fourier transform of f (in the sense of distributions).

Let M denote the Fourier transform of $m(x) = e^{-|x|}$.

Diff. Equation, $f'' - 4 . f = m$

Apply Fourier transform on both sides:

$$(2\pi i s)^2 . F(s) - 4 . F(s) = M(s)$$

$$(-4\pi^2 s^2 - 4) . F(s) = M(s)$$

From table: $M(s) = \dfrac{2}{4\pi^2 s^2 + 1}$

$$-4(1 + \pi^2 s^2) . F(s) = \frac{2}{4\pi^2 s^2 + 1}.$$

Step 2. Solve the algebraic equation:

$$-4\left(1+\pi^2 s^2\right)\cdot F(s) = \frac{2}{4\pi^2 s^2+1}$$

Real roots of: $-4\left(1+\pi^2 s^2\right) = 0$.

NO REAL ROOT. $\Rightarrow$ General Solution of.

$-4\left(1+\pi^2 s^2\right)\cdot F(s) = 0$

is $F(s) = 0$.

$$\Rightarrow \quad F(s) = \frac{2}{-4\left(1+\pi^2 s^2\right)\left(4\pi^2 s^2+1\right)} = -\frac{1}{2\left(1+\pi^2 s^2\right)\left(1+4\pi^2 s^2\right)}$$

Step 3 Inverse Fourier Transform:

$$-\frac{1}{2\left(1+\pi^2 s^2\right)\left(1+4\pi^2 s^2\right)} = \frac{A}{1+\pi^2 s^2} + \frac{B}{1+4\pi^2 s^2}$$

$$-\frac{1}{2} = \underbrace{A+B}_{-\frac{1}{2}} + \underbrace{(4A+B)}_{0}\pi^2 s^2$$

$$\begin{cases} A+B = -\frac{1}{2} \\ 4A+B = 0 \end{cases} \longrightarrow \begin{array}{l} -3A = -\frac{1}{2} \Rightarrow A = \frac{1}{6} \\ B = -4A \qquad\qquad B = -\frac{4}{6} = -\frac{2}{3} \end{array}$$

Thus:

$$F(s) = \frac{1}{6}\,\frac{1}{1+\pi^2 s^2} - \frac{2}{3}\,\frac{1}{1+4\pi^2 s^2}$$

Inverse Fourier transform:

$$M(s) = \frac{2}{1+4\pi^2 s^2} \xrightarrow{\text{I.F.T.}} e^{-|x|} = m(x).$$

$$M_1(s) = M\left(\frac{s}{2}\right) = \frac{2}{1+\pi^2 s^2} \xrightarrow{\text{I.F.T.}} 2\cdot m(2x) = 2\, e^{-2|x|}$$

$$F(s) = \frac{1}{12} M_1(s) - \frac{1}{3} M(s) \xrightarrow{\text{IFT}} \cancel{f(x) = \frac{1}{12} m(x) - \frac{2}{3} m(2x)} \quad \cancel{= \frac{1}{12} e^{-|x|} - \frac{2}{3} e^{-2|x|}}.$$

$$f(x) = \frac{1}{12}\, 2\cdot m(2x) - \frac{1}{3}\, m(x) =$$

$$= \frac{1}{6} e^{-2|x|} - \frac{1}{3} e^{-|x|}.$$

Thus: $f(x) = \frac{1}{6} e^{-2|x|} - \frac{1}{3} e^{-|x|}$

<u>Method 2</u> [Uses the Green's Function].

~~(Start with: $f'' - 4\cdot f = \delta$).~~

<u>Step 1</u>. Construct: $g'' - 4g = \delta$.

The function (distribution) g is called the $\boxed{\text{Green's Function}}$.

How: Fourier transform:

Let G denote the F.T. of g : $(2\pi i s)^2 \cdot G(s) - 4\cdot G(s) = 1.$

$$\underbrace{\left(-4\pi^2 s^2 - 4\right)}_{\text{no real roots}} \cdot G(s) = 1$$

$$\Rightarrow \quad G(s) = \frac{1}{-4\pi^2 s^2 - 4} = -\frac{1}{4}\,\frac{1}{1+\pi^2 s^2}$$

Inverse Fourier transform:

$$g(x) = -\frac{1}{8}\,2\,e^{-2|x|} = -\frac{1}{4}\,e^{-2|x|}$$

$\to$ Green's Function.

Step 2.

$\to m(x) = e^{-|x|}$

Let $f = g * m$.

Then: $f'' - 4\cdot f = (g*m)'' - 4\cdot(g*m) =$

$= g'' * m - (4g)*m = \underbrace{(g'' - 4g)}_{\delta} * m =$

$= \delta * m = m.$

$\longrightarrow$ The general solution of $f'' - 4\cdot f = m$

is $f = g * m$.

$$f(x) = \int_{-\infty}^{\infty} g(x-y)\,m(y)\,dy = \int_{-\infty}^{\infty} -\frac{1}{4} e^{-2|x-y|} \cdot e^{-|y|}\,dy$$

$$= \dots = \frac{1}{6} e^{-2|x|} - \frac{1}{3} e^{-|x|}.$$

<u>Example</u>. Solve: $f'' + f = x^2$

in the space of distributions.

<u>Solution</u>:

1. Let F denote the Fourier transform of f.

Let $m(x) = x^2$. Let $M(s)$ denote the Fourier transform of m.

Use table: $1 \xrightarrow{F.T.} \delta(s)$

$$m(x) = x^2 \cdot 1 \xrightarrow{F.T.} \underbrace{\frac{1}{(-2\pi i)^2} \cdot \frac{d^2}{ds^2} \delta(s)}_{M(s)} = -\frac{1}{4\pi^2} \delta''$$

Equation becomes:

$$(2\pi i s)^2 \cdot F(s) + F(s) = -\frac{1}{4\pi^2} \delta''(s)$$

$$(1 - 4\pi^2 s^2) \cdot F(s) = -\frac{1}{4\pi^2} \delta''(s)$$

2. $A(s) = 1 - 4\pi^2 s^2 = (1 - 2\pi s) \cdot (1 + 2\pi s) \implies$ Real Roots: $s_1 = \frac{1}{2\pi}$, $s_2 = -\frac{1}{2\pi}$

General Solution:

$$F(s) = \underbrace{-\frac{1}{4\pi^2 (1 - 4\pi^2 s^2)} \cdot \delta''(s)}_{\text{Simplify:}} + c_1 \cdot \delta\left(s - \frac{1}{2\pi}\right) + c_2 \cdot \delta\left(s + \frac{1}{2\pi}\right)$$

How to simplify : see Page (1).

$$h_1(x)\cdot\delta(x) = h_1(0)\cdot\delta(x).$$

$$h_2(x)\cdot\delta'(x) = -h_2'(0)\cdot\delta(x) + h_2(0)\cdot\delta'(x).$$

$$h_3(x)\cdot\delta''(x) = h_3''(0)\cdot\delta(x) - 2h_3'(0)\cdot\delta'(x) + h_3(0)\cdot\delta''(x)$$

$$\int_{-\infty}^{\infty} h_2(x)\cdot\delta'(x)\cdot\phi(x)\,dx = -(h_2\cdot\phi)'\Big|_{x=0} = -h_2'(0)\cdot\phi(0) - h_2(0)\cdot\phi'(0)$$

$$\int_{-\infty}^{\infty} h_3(x)\cdot\delta''(x)\,\phi(x)\,dx = (h_3\cdot\phi)''\Big|_{x=0} = h_3''(0)\cdot\phi(0) + 2h_3'(0)\phi'(0) + h_3(0)\phi''(0)$$

Apply this formula for $h_3(s) = -\dfrac{1}{4\pi^2(1-4\pi^2s^2)}$

$$h_3(0) = -\frac{1}{4\pi^2}$$

$$h_3'(s) = \frac{-8\pi^2\cdot s}{4\pi^2(1-4\pi^2s^2)^2} \longrightarrow h_3'(0) = 0.$$

$$h_3''(s) = \frac{-8\pi^2(1-4\pi^2s^2)^2 - (-8\pi^2s)\cdot 2(4\pi^2s^2-1)\cdot 8\pi^2 s}{4\pi^2\ (1-4\pi^2s^2)^4}$$

$$h_3''(0) = \frac{-8\pi^2}{4\pi^2} = -2$$

$$\Rightarrow -\frac{1}{4\pi^2(1-4\pi^2s^2)}\delta''(s) = -2\cdot\delta(x) - \frac{1}{4\pi^2}\delta''(x)$$

$$\Rightarrow F(s) = -2\delta(s) - \frac{1}{4\pi^2}\delta''(s) + c_1\,\delta\left(s-\frac{1}{2\pi}\right) + c_2\,\delta\left(s+\frac{1}{2\pi}\right)$$

3. Inverse Fourier transform:

$$f(x) = -2 + x^2 + c_1 e^{ix} + c_2 e^{-ix}$$

or:

$$f(x) = x^2 - 2 + A_1 \cdot \cos(x) + A_2 \cdot \sin(x)$$

$A_1, A_2 \in \mathbb{C}$.