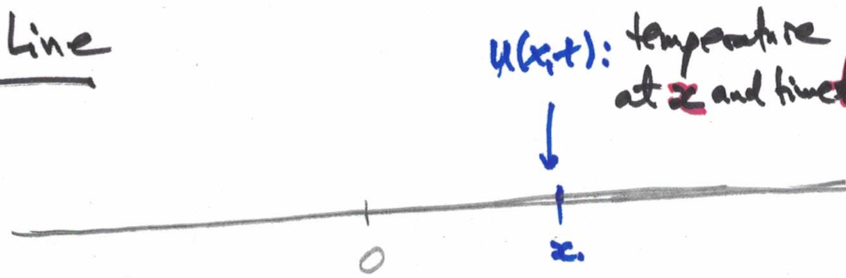
The Heat Equation

Heat Equation on the Real Line



$$\begin{cases} \dfrac{\partial u}{\partial t} = a \cdot \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = \varphi(x). \end{cases} \quad , -\infty < x < +\infty. \quad (x \in \mathbb{R})$$

initial temperature field: At time $t=0$, the distribution of temperature is $\varphi(x)$.

<u>Given:</u> $a>0$, $\varphi = \varphi(x)$ (initial condition).

<u>Want:</u> $u = u(x,t)$, $t \geq 0$.

<u>Step 1</u>. Let $\hat{u}(s,t)$ denote the Fourier transform of the function $x \longmapsto u(x,t)$.

$$\hat{u}(s,t) = \int_{-\infty}^{\infty} e^{-2\pi i s x} \, u(x,t) \, dx$$

$$u(x,t) = \int_{-\infty}^{\infty} e^{2\pi i s x} \, \hat{u}(s,t) \, ds$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} e^{2\pi i s x} \, \hat{u}(s,t) \, ds = \int_{-\infty}^{\infty} e^{2\pi i s x} \, \frac{\partial \hat{u}}{\partial t}(s,t) \, ds.$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \int_{-\infty}^{\infty} e^{2\pi i s x} \, \hat{u}(s,t) \, ds = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial x} e^{2\pi i s x} \right) \hat{u}(s,t) \, ds = \int_{-\infty}^{\infty} e^{2\pi i s x} \cdot 2\pi i s \, \hat{u}(s,t$$

$$\underline{\frac{\partial^2 u}{\partial x^2}} = \cdots = \int_{-\infty}^{\infty} e^{2\pi i s x} \cdot (2\pi i s)^2 \, \hat{u}(s,t)\, ds$$

PDE : $\dfrac{\partial u}{\partial t} = a \dfrac{\partial^2 u}{\partial x^2}$, $u_t = a \cdot u_{xx}$: equivalent equation: different convention of writing partial derivatives.

$$\int_{-\infty}^{\infty} e^{2\pi i s x} \frac{\partial \hat{u}}{\partial t}(s,t)\, ds = \int_{-\infty}^{\infty} e^{2\pi i s x} \cdot (2\pi i s)^2 a \, \hat{u}(s,t)\, ds.$$

for all $t \geq 0$
for all $\underline{x \in \mathbb{R}}$.

$$\Rightarrow \quad \frac{\partial \hat{u}}{\partial t}(s,t) = (2\pi i s)^2 a \, \hat{u}(s,t), \quad \text{for all } s \in \mathbb{R}$$
for all $t \geq 0$.

$$\boxed{\frac{\partial \hat{u}}{\partial t} = -4\pi^2 a s^2 \cdot \hat{u}(s,t).}$$

Step 2.
<u>Fix</u> $s \in \mathbb{R}$. Let $f : t \longmapsto f(t) = \hat{u}(s,t)$

$$f' = -4\pi^2 a s^2 \cdot f$$

$$f' + \underbrace{(4\pi^2 s^2 a)}_{\text{independent of } t} \cdot f = 0.$$

$f(t) = e^{rt}$, $\quad r e^{rt} + (4\pi^2 s^2 a) e^{rt} = 0 \Rightarrow r = -4\pi^2 s^2 a.$

$f(t) = C \cdot e^{-4\pi^2 s^2 a t}$, where C is independent of t.

We obtained:

$$\hat{u}(s,t) = C(s) \cdot e^{-4\pi^2 s^2 a t}$$

The function $C(s)$ is going to be computed from initial condition.

At $t=0$: $u(x,0) = \varphi(x)$ $\mapsto$ C is the Fourier transform of φ

$$\hat{u}(s,0) = C(s).$$

$$C(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} \varphi(x)\,dx. = \hat{\varphi}(s)$$

$$\boxed{\hat{u}(s,t) = \hat{\varphi}(s) \cdot e^{-4\pi^2 s^2 a t}}$$

Step 3. Inverse Fourier transform $\hat{u}$.

$$u(x,t) = \int_{-\infty}^{\infty} e^{2\pi i s x} \cdot \hat{\varphi}(s) \cdot e^{-4\pi^2 s^2 a t}\, ds$$

Since $\hat{u}$ is the product of $\hat{\varphi}$ with. $g(s) = e^{-4\pi^2 s^2 a t}$

then $u = \varphi * (\text{inverse Fourier transform of } g)$.

Inverse Fourier Transform of g:

$$g_0(s) = e^{-\pi s^2} \xrightarrow{\text{IFT}} e^{-\pi x^2}$$

$$g(s) = g_0(\sqrt{4\pi a t} \cdot s) = e^{-4\pi^2 a t s^2} \xrightarrow{\text{IFT}} \gamma(x) = \frac{1}{\sqrt{4\pi a t}} e^{-\frac{\pi x^2}{4\pi a t}}$$

$$\gamma(x) = \frac{1}{\sqrt{4\pi a t}} \exp\left(-\frac{x^2}{4at}\right)$$: γ depends on x and t.

(4).

Therefore:

$$\boxed{u(x,t) = (\varphi * \gamma)(x) = \int_{-\infty}^{\infty} \gamma(x-y)\cdot\varphi(y)\,dy = \frac{1}{\sqrt{4\pi a t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4at}}\cdot\varphi(y)\,dy.}$$

- γ: the heat kernel.

At happens at $t=0$:

$$u(x,t) = \int_{-\infty}^{\infty} e^{2\pi i s x}\,\hat{\varphi}(s)\,e^{-4\pi^2 s^2 a t}\,ds$$

$$\to u(x,0) = \int_{-\infty}^{\infty} e^{2\pi i s x}\,\hat{\varphi}(s)\cdot 1\,ds = \varphi(x).$$

But also:

$$\lim_{t\downarrow 0} u(x,t) = \lim_{t\downarrow 0}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi a t}}\,e^{-\frac{(x-y)^2}{4at}}\,\varphi(y)\,dy = \varphi(x).$$

$$\int_{-\infty}^{\infty}\delta(x-y)\,\varphi(y)\,dy = \varphi(x)$$

For φ test functions (in $\mathcal{S}$).

$$\boxed{\lim_{t\downarrow 0}\frac{1}{\sqrt{4\pi a t}}\,e^{-\frac{x^2}{4at}} = \delta(x).}$$

(Should be understood in weak sense).

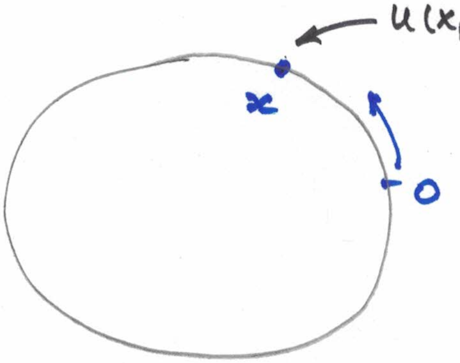
The meaning:

$$\forall\varphi\in\mathcal{S},\quad \underbrace{\lim_{t\downarrow 0}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi a t}}\,e^{-\frac{x^2}{4at}}\,\varphi(x)\,dx. = \varphi(0)} = \delta\{\varphi\} = \int_{-\infty}^{\infty}\delta(x)\varphi(x)dx$$

Heat Equation on $\mathbb{T}^1$, or Heat Equation with periodic boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial t} = a \dfrac{\partial^2 u}{\partial x^2} & \longleftarrow \text{Diff. Equ.} \\ u(x,0) = \varphi(x) & \longleftarrow \text{Initial Condition.} \\ u(x+p,t) = u(x,t). & \longleftarrow \text{Periodic Condition.} \end{cases}$$

Circle of circumference p.



$\longleftarrow u(x,t)$: temperature at position x and time t.

Step 1 Expand $x \longmapsto u(x,t)$ in a Fourier Series.

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t)\, e^{\frac{2\pi i n x}{p}} \quad , \quad c_n(t) = \frac{1}{p}\int_0^p e^{-\frac{2\pi i n x}{p}} u(x,t)\,dx$$

$$\frac{\partial u}{\partial t} = \sum_{n=-\infty}^{\infty} \overset{\bullet}{C}_n(t) \cdot e^{\frac{2\pi i n x}{p}} \qquad \overset{\bullet}{} \leftarrow \frac{d}{dt}$$

$$\frac{\partial u}{\partial x} = \sum_{n=-\infty}^{\infty} C_n(t) \cdot \frac{2\pi i n}{p}\, e^{\frac{2\pi i n x}{p}}$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_n c_n(t) \left(\frac{2\pi i n}{p}\right)^2 e^{\frac{2\pi i n x}{p}}$$

$$\Rightarrow \sum_n \dot{c}_n\, e^{\frac{2\pi i n x}{p}} = \sum_n c_n \frac{-4\pi^2 n^2 a}{p^2} \cdot e^{\frac{2\pi i n x}{p}} \quad , \quad \forall x, \forall t \geq 0.$$

$\Rightarrow$ Fourier basis:

$$\dot{c}_n$$

For every $n \in \mathbb{Z}$, $t \geq 0$: $\dfrac{dc_n}{dt} = -\dfrac{4\pi^2 n^2 a}{p^2} \cdot c_n.$

Step 2. Solve $\dfrac{dc_n}{dt} = -\dfrac{4\pi^2 n^2 a}{p^2} c_n.$

$$c_n(t) = c_n(0) \cdot e^{-\frac{4\pi^2 n^2 a t}{p^2}}$$

Initial condition: $\varphi(x) = u(x,0) = \sum_{n=-\infty}^{\infty} c_n(0)\, e^{\frac{2\pi i n x}{p}}$

$$\Rightarrow \underbrace{c_n(0) = \frac{1}{p}\int_0^p e^{-\frac{2\pi i n x}{p}} \varphi(x)\, dx.}$$

Step 3: Solution:

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n(0)\, e^{-\frac{4\pi^2 n^2 a t}{p^2}} \cdot e^{\frac{2\pi i n x}{p}} =$$

$$= \frac{1}{p} \cdot \sum_{n=-\infty}^{\infty} \int_0^p e^{-\frac{2\pi i n y}{p}} \varphi(y)\, dy \cdot e^{\frac{2\pi i n x}{p}} \cdot e^{-\frac{4\pi^2 n^2 a t}{p^2}} =$$

$$= \cancel{\tfrac{1}{p}} \int_0^p \Big[\underbrace{\frac{1}{p} \sum_{n=-\infty}^{\infty} e^{-\frac{4\pi^2 n^2 a t}{p^2} + \frac{2\pi i n (x-y)}{p}}}_{H(t,\,x-y)\ :\ \text{discrete Heat Kernel}} \Big] \varphi(y)\, dy.$$

$$\boxed{u(x,t) = \int_0^p H(t, x-y)\, \varphi(y)\, dy}$$

The Wave Equation

Wave Equation on the Real Line:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = \varphi_0(x) \quad , \quad -\infty < x < \infty \\ \dfrac{\partial u}{\partial t}(x,0) = \varphi_1(x), \end{cases}$$

$c > 0$
↳ the wave speed.

Wave Equation on $\mathbb{T}^1$, or with Periodic Boundary Condition.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = \varphi_0(x) \\ \dfrac{\partial u}{\partial t}(x,0) = \varphi_1(x) \\ u(x+p,t) = u(x,t), \end{cases}$$

← Diff. Eqn.

} I.C. initial Condition.

$0 \leq x < p$

$\forall t$.

Given: c^2, $p > 0$

~~$p>0$~~

↳ Periodic Boundary Condition.

Solution:

$$\boxed{u(x,t) = \frac{1}{2}\left[\varphi_0(x+ct) + \varphi_0(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \varphi_1(y)\,dy.}$$

Assume: $\frac{\partial u}{\partial t}(x,0) = \varphi_1(x) = 0.$

$$u(x,t) = \frac{1}{2}\left[\varphi_0(x+ct) + \varphi_0(x-ct)\right].$$