L26

Windowed Fourier Transform (2)

Examples:

Consider $g(x) = e^{-\pi x^2} \dashrightarrow$ window.

$f(x) = \sin(20\pi x)$.

Problem: $V_g f = ?$

$$V_g f(t,\omega) = \int_{-\infty}^{\infty} e^{-2\pi i \omega x} \underbrace{\sin(20\pi x)}_{= \frac{e^{20\pi i x} - e^{-20\pi i x}}{2i}} e^{-\pi(x-t)^2} dx =$$

$$= \frac{1}{2i} \int_{-\infty}^{\infty} e^{-2\pi i x(\omega - 10)} e^{-\pi(x-t)^2} dx - \frac{1}{2i} \int_{-\infty}^{\infty} e^{-2\pi i x(\omega+10)} \cdot e^{-\pi(x-t)^2} dx$$

$u = x - t$, $x = u + t$

$$= \frac{1}{2i} e^{-2\pi i t(\omega-10)} \underbrace{\int_{-\infty}^{\infty} e^{-2\pi i u(\omega-10)} e^{-\pi u^2} du}_{\text{F.T. of unit Gaussian at } \omega - 10} - \frac{1}{2i} e^{-2\pi i t(\omega+10)} \underbrace{\int_{-\infty}^{\infty} e^{-2\pi i u(\omega+10)} e^{-\pi u^2} du}_{\text{F.T. of unit Gaussian at } \omega+10}$$

$$= \frac{1}{2i} e^{-2\pi i t(\omega-10)} e^{-\pi(\omega-10)^2} - \frac{1}{2i} e^{-2\pi i t(\omega+10)} e^{-\pi(\omega+10)^2}.$$

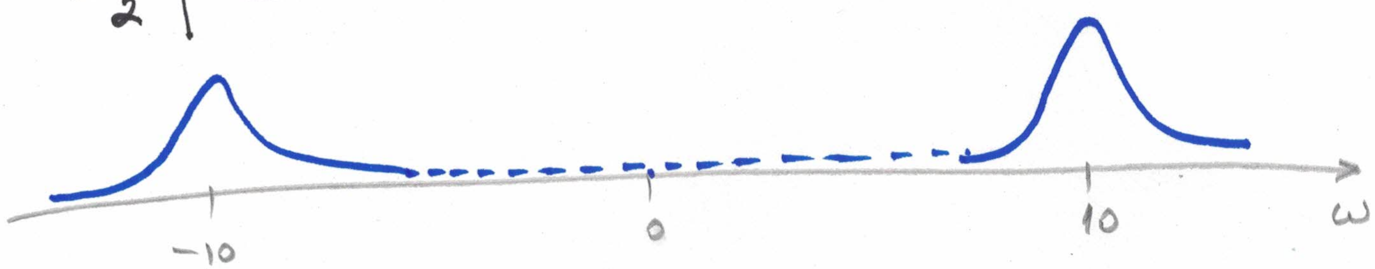
Thus:

$$V_g f(t,\omega) = \frac{1}{2i} e^{-2\pi i t(\omega-10)} \cdot e^{-\pi(\omega-10)^2} - \frac{1}{2i} e^{-2\pi i t(\omega+10)} \cdot e^{-\pi(\omega+10)^2}$$

Plot the magnitude of $|V_g f|$ as function of (t,ω).

$$|V_g f(t,\omega)| = \left| \frac{1}{2i} e^{-2\pi i t\omega} \cdot \left(e^{+20\pi i t} e^{-\pi(\omega-10)^2} - e^{-20\pi i t} \cdot e^{-\pi(\omega+10)^2} \right) \right|$$

$$= \frac{1}{2} \left| e^{20\pi i t} \underbrace{e^{-\pi(\omega-10)^2}} - e^{-20\pi i t} \underbrace{e^{-\pi(\omega+10)^2}} \right|$$



At $\omega = 0$:

$$e^{-\pi(\omega-10)^2}\Big|_{\omega=0} = e^{-100\pi} \simeq e^{-314} \simeq 4\cdot 10^{-137}$$

$$e^{-\pi(\omega+10)^2}\Big|_{\omega=0} = e^{-100\pi} \simeq e^{-314} \simeq 4\cdot 10^{-137} \Bigg\}\Bigg\} = 0.$$

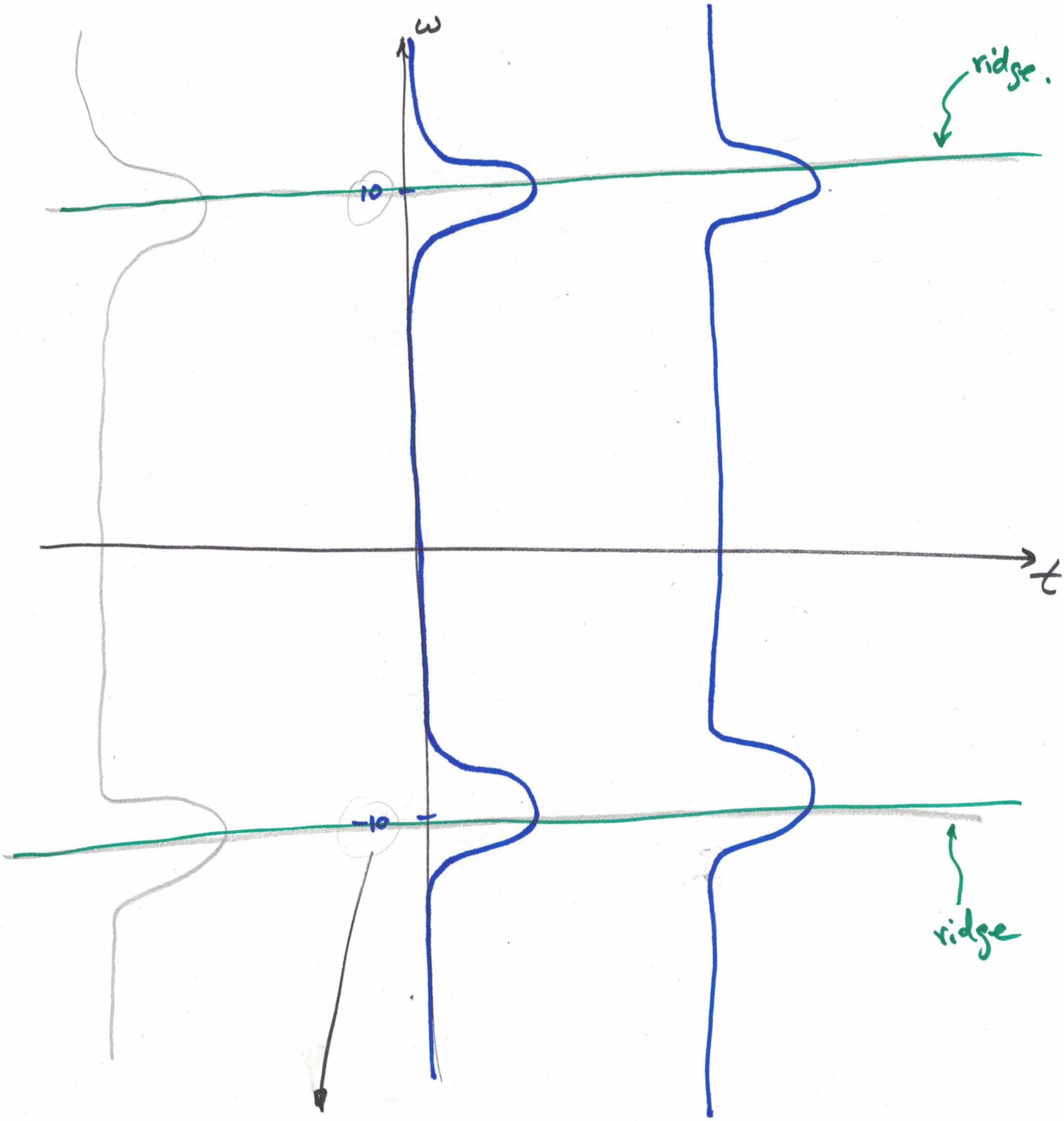
We can approximate both values at $\omega = 0$ by 0.

Valid approximations: $e^{-\pi(\omega-10)^2} \cdot e^{-\pi(\omega+10)^2} = 0.$

$$\Rightarrow \left| a \cdot e^{-\pi(\omega-10)^2} + b\, e^{-\pi(\omega+10)^2} \right| = \left| a e^{-\pi(\omega-10)^2} \right| + \left| b e^{-\pi(\omega+10)^2} \right|$$

$$= |a| \cdot e^{-\pi(\omega-10)^2} + |b|\, e^{-\pi(\omega+10)^2}$$

$$\Rightarrow \quad |V_g f(t,\omega)| \approx \frac{1}{2} e^{-\pi(\omega-10)^2} + \frac{1}{2} e^{-\pi(\omega+10)^2}$$



related: $f(x) = \sin(20\pi x) = \sin(2 \cdot 10 \cdot \pi x)$

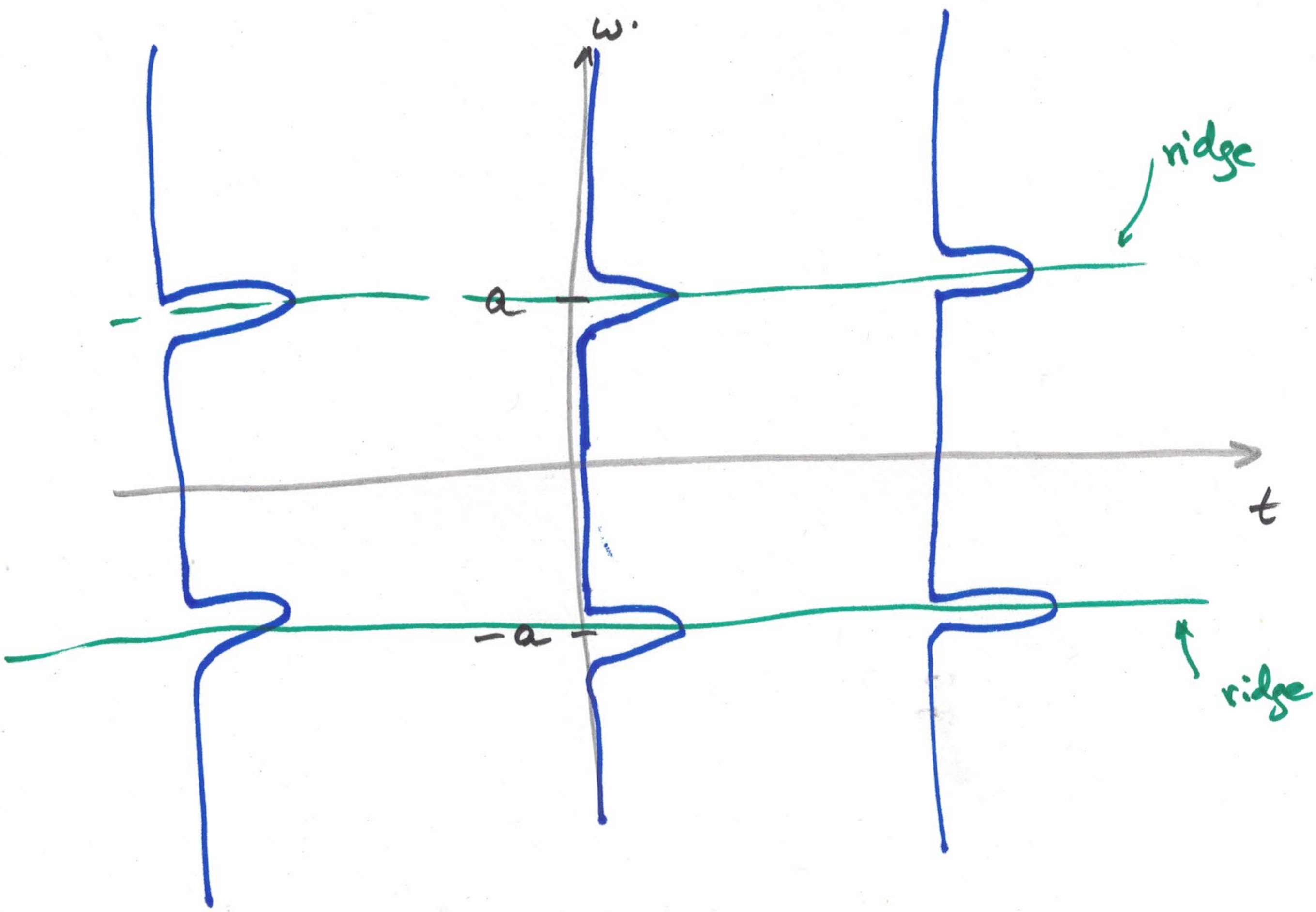
If $f(x) = \sin(2a\pi x)$, $a > 0$.

$$\Rightarrow V_g f(t,\omega) = \frac{1}{2i} e^{-2\pi i t(\omega - a)} e^{-\pi(\omega-a)^2} - \frac{1}{2i} e^{-2\pi i t(\omega+a)} e^{-\pi(\omega+a)^2}$$

If a is "sufficiently" large: $e^{-\pi a^2} = e^{-\pi(\omega-a)^2}\Big|_{\omega=0} \ll 1$.

$$|V_g f(t,\omega)| \approx \frac{1}{2} e^{-\pi(\omega-a)^2} + \frac{1}{2} e^{-\pi(\omega+a)^2}$$

The plot of $|V_g f(t,\omega)|$ is called the spectrogram.



Let's verify the inversion Formula:

Given $F(t,\omega) = \frac{1}{2i} e^{-2\pi i t(\omega-10)} e^{-\pi(\omega-10)^2} - \frac{1}{2i} e^{-2\pi i t(\omega+10)} e^{-\pi(\omega+10)^2}$

as windowed Fourier transform of some signal f w.r.t. window

$$g(x) = e^{-\pi x^2},$$

Want: Recover / Find. f (Do: Inverse Windowed Fourier Transf

Recall the inversion formula:

$$f(x) = \frac{1}{\|g\|^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i \omega x} F(t,\omega)\, g(x-t)\, dt\, d\omega$$

$$\|g\|^2 = \int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} e^{-2\pi x^2} dx = \frac{1}{\sqrt{2}} \underbrace{\int_{-\infty}^{\infty} e^{-\pi u^2} du}_{=1 = \hat{e^{-\pi s^2}}\big|_{s=0}} = \frac{1}{\sqrt{2}}$$

$\sqrt{2}x = u.$

$dx = \frac{1}{\sqrt{2}} du$

$$f(x) = \sqrt{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i \omega x} \left[\frac{1}{2i} e^{-2\pi i t(\omega-10)} \cdot e^{-\pi(\omega-10)^2} - \frac{1}{2i} e^{-2\pi i t(\omega+10)} e^{-\pi(\omega+10)^2} \right] e^{-\pi(x-t)^2} dt d\omega$$

$$= \frac{1}{\sqrt{2}i} \int_{-\infty}^{\infty} e^{20\pi i t} e^{-\pi(x-t)^2} \int_{-\infty}^{\infty} e^{2\pi i \omega(x-t)} e^{-\pi(\omega-10)^2} d\omega\, dt -$$

$$- \frac{1}{\sqrt{2}i} \int_{-\infty}^{\infty} e^{-20\pi i t} e^{-\pi(x-t)^2} \int_{-\infty}^{\infty} e^{2\pi i \omega(x-t)} e^{-\pi(\omega+10)^2} d\omega\, dt$$

First Inner integral: $\omega \longmapsto s$, $s = \omega - 10$, $\omega = s + 10$, $ds = d\omega$ || Second int. $s = \omega + 10$ (6)

$$f(x) = \frac{1}{\sqrt{2}i} \int_{-\infty}^{\infty} e^{20\pi i t} e^{-\pi(x-t)^2} e^{20\pi i(x-t)} \left[\int_{-\infty}^{\infty} e^{2\pi i s(x-t)} e^{-\pi s^2} ds \right] dt -$$

$$- \frac{1}{\sqrt{2}i} \int_{-\infty}^{\infty} e^{-20\pi i t} e^{-\pi(x-t)^2} e^{-20\pi i(x-t)} \underbrace{\left[\int_{-\infty}^{\infty} e^{2\pi i s(x-t)} e^{-\pi s^2} ds \right]}_{e^{-\pi(x-t)^2}} dt =$$

$$= \frac{1}{\sqrt{2}i} \int_{-\infty}^{\infty} e^{-\pi(x-t)^2 \cdot 2} \cdot e^{20\pi i x} dt - \frac{1}{\sqrt{2}i} \int_{-\infty}^{\infty} e^{-\pi(x-t)^2 \cdot 2} \cdot e^{-20\pi i x} dt =$$

$v = t - x$

$$= \frac{1}{\sqrt{2}i} e^{20\pi i x} \underbrace{\int_{-\infty}^{\infty} e^{-2\pi v^2} dv}_{= \|g\|^2 = \frac{1}{\sqrt{2}}} - \frac{1}{\sqrt{2}i} e^{-20\pi i x} \underbrace{\int_{-\infty}^{\infty} e^{-2\pi v^2} dv}_{= \|g\|^2 = \frac{1}{\sqrt{2}}} =$$

$$= \frac{1}{2i} e^{20\pi i x} - \frac{1}{2i} e^{-20\pi i x} = \underline{\underline{\sin(20\pi x)}}.$$