L27 Today:

Topic 1: Wavelet Transform

~~Topic 2: Uncertainty Inequality~~

The Continuous Wavelet Transform

Recall from examples of Windowed Fourier Transform (WFT)

$$f(x) = e^{2\pi i \omega_1 x} + e^{2\pi i \omega_2 x} \quad (\text{or, } f(x) = \sin(2\pi\omega_1 x) + \sin(2\pi \omega_2 x)).$$

and a window $g(x) = e^{-\pi x^2}$.

$\Rightarrow$ Spectrogram, $|V_g f(t,\omega)|$ exhibits two separated ridges (for $\omega > 0$)

one for $f_1(x) = e^{2\pi i \omega_1 x}$ and another for $f_2(x) = e^{2\pi i \omega_2 x}$

if and only if $|\omega_1 - \omega_2|$ is large enough.

On the other hand, if we perform the WFT with a different window g then the minimum separation $|\omega_1 - \omega_2|$ may be different.

A different transform was introduced:

Definition. Fix a function $\psi : \mathbb{R} \to \mathbb{R}$ called wavelet. The continuous wavelet transform (CWT) of a function (signal) $f : \mathbb{R} \to \mathbb{C}$ with respect to ψ, denoted $W_\psi f$ is the function,

$$W_\psi f : (\mathbb{R} \setminus \{0\}) \times \mathbb{R} \longrightarrow \mathbb{C}, \quad W_\psi f(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(x) \cdot \psi\left(\frac{x-b}{a}\right) dx$$

$$W_\psi f(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(x) \cdot \psi\left(\frac{x-b}{a}\right) dx. \quad \to \text{CWT}$$

Compare to:

$$V_g f(t,\omega) = \int_{-\infty}^{\infty} e^{-2\pi i \omega x} f(x)\, g(x-t)\, dx. \quad \to \text{WFT}$$

If $\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right)$, $g_{\omega,t}(x) = e^{2\pi i \omega x} g(x-t)$

Then:

$$W_\psi f(a,b) = \langle f, \psi_{a,b} \rangle, \quad V_g f(t,\omega) = \langle f, g_{\omega,t} \rangle.$$

You can check:

$$\int_{-\infty}^{\infty} |\psi_{a,b}(x)|^2 dx = \frac{1}{|a|} \int_{-\infty}^{\infty} \left|\psi\left(\tfrac{x-b}{a}\right)\right|^2 dx = \int_{-\infty}^{\infty} |\psi(y)|^2 dy = \|\psi\|_2^2$$

$$\int_{-\infty}^{\infty} |g_{\omega,t}(\omega)|^2 dx = \int_{-\infty}^{\infty} |g(x-t)|^2 dx = \int_{-\infty}^{\infty} |g(x)|^2 dx = \|g\|_2^2$$

For the CWT: $a \to$ Scale. (scale shift) $\dashrightarrow$ dilation.

$b \to$ time. (time shift) $\dashrightarrow$ translation.

$$G = \{(a,b) : a,b \in \mathbb{R},\ a \neq 0\}. \dashrightarrow \left(U(a,b)\,\psi\right)(x) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right).$$

U: unitary operator

$\hookrightarrow$ Form a group, $(G,*)$ $\to$ NOT A COMMUTATIVE GROUP!

$$(a,b) = (a_1,b_1)*(a_2,b_2) \leftrightarrow U(a,b) = U(a_1,b_1)\cdot U(a_2,b_2)$$

$\Longrightarrow$ The Representation Theory of the $ax+b$ group.

$$\Big(U(a_1,b_1)\underbrace{U(a_2,b_2)\psi}_{\varphi}\Big)(x) = \big(U(a_1,b_1)\varphi\big)(x) = \frac{1}{\sqrt{|a_1|}}\varphi\left(\frac{x-b_1}{a_1}\right) =$$

$$= \frac{1}{\sqrt{|a_1|}}\frac{1}{\sqrt{|a_2|}}\psi\left(\frac{\frac{x-b_1}{a_1}-b_2}{a_2}\right) = \frac{1}{\sqrt{|a_1a_2|}}\psi\left(\frac{x-b_1-a_1b_2}{a_1a_2}\right) =$$

$$= \frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right).$$

Hence: $a = a_1a_2$, $b = b_1 + a_1b_2$

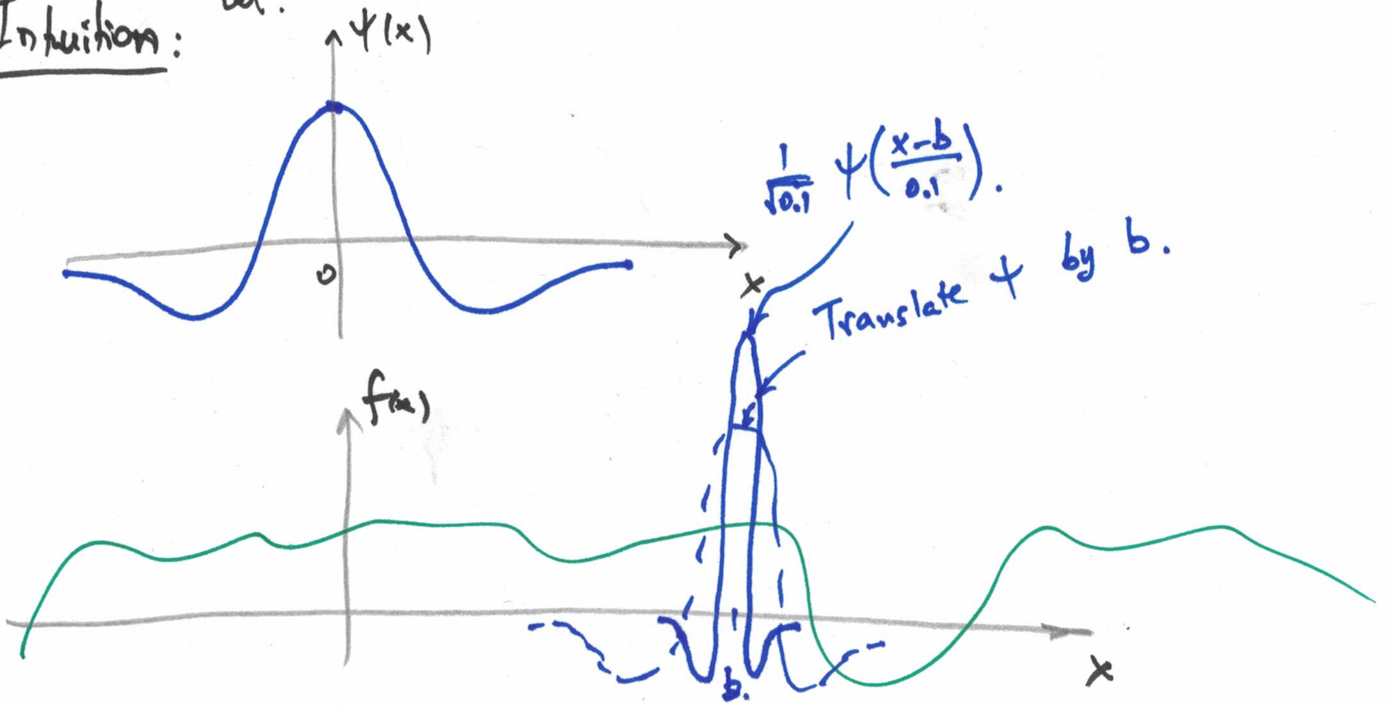
$\underline{(a_1,b_1)*(a_2,b_2) = (a_1a_2,\ b_1+a_1b_2)}$

$(a,b)*(1,0) = (a\cdot 1,\ b+a\cdot 0) = (a,b)$

$(1,0)*(a,b) = (1\cdot a,\ 0+1\cdot b) = (a,b)$

$$\Rightarrow (a,b)*(1,0) = (1,0)*(a,b) = (a,b).$$

<u>Intuition</u>: Let:



How to invert the CWT:

Theorem. Assume $\psi : \mathbb{R} \to \mathbb{R}$ the wavelet satisfies:

(1) $\displaystyle\int_{-\infty}^{\infty} |\psi(x)|^2\,dx < \infty$, $\psi \in L^2(\mathbb{R})$

(2) $\displaystyle C_\psi := \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\,d\omega < \infty.$ $\xrightarrow{\text{If } \hat{\psi} \text{ is continuous.}}$ $\hat{\psi}(0) = 0 \dashrightarrow \displaystyle\int_{-\infty}^{\infty} \psi(x)\,dx =$

"wavelet"

Then:

(1) If $f_1, f_2 \in L^2(\mathbb{R})$, let $W_\psi f_1$, $W_\psi f_2$ denote their CWT respectively. Then:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_\psi f_1(a,b) \cdot \overline{W_\psi f_2(a,b)}\ \underbrace{\frac{1}{a^2}\,db\,da}\ = C_\psi \cdot \int_{-\infty}^{\infty} f_1(x) \cdot \overline{f_2(x)}\,dx$$

or,

$$\left\langle W_\psi f_1,\ W_\psi f_2 \right\rangle_{L^2\left(\mathbb{R}^2,\ \frac{1}{a^2}\,da\,db\right)} = C_\psi \cdot \langle f_1, f_2 \rangle.$$

(Generalized Plancherel / Parseval Identity).

2) Let $f \in L^2(\mathbb{R})$ and. $F(a,b) = W_\psi f(a,b)$. Then:

$$f(x) = \frac{1}{C_\psi} \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(a,b) \cdot \psi_{a,b}(x)\, \frac{1}{a^2}\,da\,db = \frac{1}{C_\psi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(a,b) \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right) \frac{da}{a}$$

in L^2-sense

<u>Why:</u>

If $f_1 = f_2 = f$. <u>Want:</u> $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |W_\psi f(a,b)|^2 \frac{da\,db}{a^2} = c_\psi \cdot \int_{-\infty}^{\infty} |f(x)|^2 dx$

Generalized (Plancherel identity)

$$W_\psi f(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(x)\, \psi\left(\frac{x-b}{a}\right) dx = \langle f, \psi_{a,b} \rangle \underset{\uparrow}{=} \langle \hat{f}, \hat{\psi}_{a,b} \rangle.$$

Plancherel-Parseval.

$$\hat{\psi}_{a,b}(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i \omega x} \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right) dx = \int_{-\infty}^{\infty} e^{-2\pi i \omega (ay+b)} \psi(y) \frac{1}{\sqrt{|a|}} |a|\, dy =$$

$$y = \frac{x-b}{a}; \quad x = ay+b$$
$$dx = |a|\, dy$$

$$= \sqrt{|a|}\, e^{-2\pi i b\omega} \int_{-\infty}^{\infty} e^{-2\pi i a \omega y} \psi(y)\, dy = \sqrt{|a|}\, e^{-2\pi i b \omega}\, \hat{\psi}(a\omega).$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |W_\psi f(a,b)|^2 \frac{da\,db}{a^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |a| \cdot \left| \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{2\pi i b \omega}\, \overline{\hat{\psi}(a\omega)}\, d\omega \right|^2 \frac{da\,db}{a^2} =$$

$$= \int_{-\infty}^{\infty} \frac{1}{|a|} \underbrace{\left[\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} e^{2\pi i b \omega} \underbrace{\left(\hat{f}(\omega) \cdot \overline{\hat{\psi}(a\omega)} \right)}_{g(\omega).} d\omega \right|^2 db. \right]}_{\|g\|_2^2} da = \int_{-\infty}^{\infty} \frac{1}{|a|} \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 |\hat{\psi}(a\omega)|^2 d\omega$$

$$= \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \left(\int_{-\infty}^{\infty} \frac{1}{|a|} |\hat{\psi}(a\omega)|^2 \, da \right) d\omega$$

__But:__

$$\int_{-\infty}^{\infty} \frac{1}{|a|} |\hat{\psi}(a\omega)|^2 \, da = \int_{-\infty}^{\infty} \frac{1}{\frac{|s|}{\cancel{|\omega|}}} |\hat{\psi}(s)|^2 \frac{1}{\cancel{|\omega|}} \, ds = \int_{-\infty}^{\infty} \frac{1}{|s|} |\hat{\psi}(s)|^2 \, ds = C_\psi$$

$$s = a \cdot \omega, \quad a = \frac{1}{\omega} s$$

$$da = \frac{1}{|\omega|} \, ds.$$

Thus:

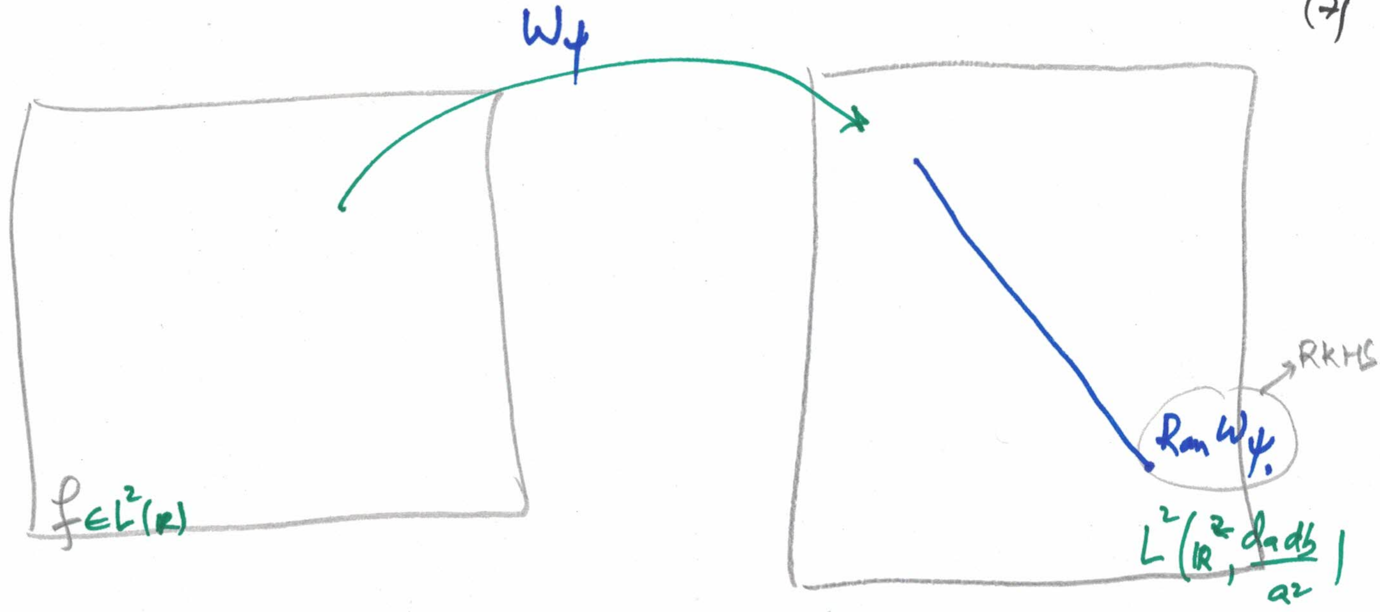
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |W_\psi f(a,b)|^2 \frac{da\,db}{a^2} = C_\psi \cdot \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = C_\psi \cdot \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

By similar arguments as for WFT (polarization identity & computing the adjoint)

$\longrightarrow$ obtain (1), (2).

$$\cancel{f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{W_\psi f(a,b)}_{\langle f, \psi_{a,b} \rangle} \psi_{a,b}(x) \frac{da\,db}{a^2} = \left\langle f, \underbrace{\frac{1}{C_\psi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_{a,b}(x) \overline{\psi_{a,b}(\cdot)} \frac{da\,db}{a^2}}_{k_x(\cdot) = k(x,\cdot)\text{ : kernel.}} \right\rangle}$$

$$\cancel{\longrightarrow f(x) = \langle f, k_x \rangle}$$ ~~$\Rightarrow$ Reproducing Kernel Hilbert Space (RKHS)~~



$$F(a,b) = W_\psi f(a,b) = \langle f, \psi_{a,b} \rangle = \left\langle \frac{1}{C_\psi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(a',b')\, \psi_{a',b'}(x) \frac{da\,db}{a^2}, \psi_{a,b} \right\rangle =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(a',b') \cdot \underbrace{\frac{1}{C_\psi} \langle \psi_{a'b'}, \psi_{ab} \rangle}_{k_{a,b}(a',b')} \frac{da\,db}{a^2} = \langle F, k_{a,b}(\cdot,\cdot) \rangle$$

RKHS.

Ran(W_ψ) is a RKHS

<u>NOTE</u>: $\varphi \in \mathcal{S} \dashrightarrow \varphi(x_0) = \int_{-\infty}^{\infty} \varphi(x) \cdot \delta(x - x_0)\, dx \equiv \langle \varphi, \delta(\cdot - x_0) \rangle$

But $\delta(\cdot - x_0)$ is <u>not</u> in the same space $\mathcal{S}$ as φ.

Here: $k_{a,b}(a',b') = \frac{1}{C_\psi} \langle \psi_{a,b}, \psi_{a',b'} \rangle \equiv$ CWT of $\frac{1}{C_\psi} \psi_{a,b} \in \mathrm{Ran}(W_\psi)$.