Fourier Series

Goals/Objectives:

$$V = L^2[0,1] = \left\{ f : [0,1] \longrightarrow \mathbb{C} \;:\; \int_0^1 |f(x)|^2 dx < \infty \right\}.$$

Scalar product:

$$\langle f, g \rangle = \int_0^1 f(x) \cdot \overline{g(x)} \, dx.$$

Theorem The set of functions $\{ e_n, n \in \mathbb{Z} \}$ is an Orthonormal Basis (ONB) for $L^2[0,1]$, where

$$e_n(x) = e^{2\pi i n x} = \cos(2\pi n x) + i \sin(2\pi n x).$$

What is an $\boxed{\text{ONB}}$ in $\left(L^2[0,1], \langle \cdot, \cdot \rangle \right)$?

Definition:

Need to satisfy:

① ORTHONORMALITY:

For every n, m +/- integer:

$$\langle e_n, e_m \rangle = \begin{cases} 1, & n = m \\ 0, & n \neq m. \end{cases}$$

Specifically:

$$\int_0^1 e_n(x) \cdot \overline{e_m(x)} \, dx = \begin{cases} 1, & n = m \\ 0, & n \neq m. \end{cases}$$

② DENSE APPROXIMATION ("SPANNING"):

Dense Approximation:

For any $f \in L^2[0,1]$, i.e., $f:[0,1] \longrightarrow \mathbb{C}$ s.t. $\int_0^1 |f(x)|^2 dx < \infty$

For any $\varepsilon > 0$,

there is $N \geq 0$ integer and scalar, $c_{-N}, c_{-N+1}, \ldots, c_{-1}, c_0, c_1, \ldots c_N$

(2N + 1 complex numbers) such that:

$$\int_0^1 \left| f(x) - \left(c_{-N} \cdot e_{-N}(x) + \cdots + c_{-1} \cdot e_{-1}(x) + c_0 e_0(x) + \cdots + c_N e_N(x) \right) \right|^2 dx <$$

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_0^1 |f(x)|^2 dx}$$

More Compact Definition.

Dense Approximation:

$$\boxed{\forall f \in L^2[0,1] \; \forall \varepsilon > 0 \; \exists N \; \exists (c_k)_{k=-N}^{N} \text{ s.t. } \left\| f - \sum_{k=-N}^{N} c_k e_k \right\|^2 < \varepsilon}$$

$$|z|^2 = z \cdot \bar{z} \;, \quad |z| = \sqrt{z \cdot \bar{z}}$$

$$= (Re(z))^2 + (Im(z))^2$$

Why:

① check ORTHONORMALITY.

$$\langle e_n, e_m \rangle = ?$$

For $\boxed{n = m}$:

$$\langle e_n, e_n \rangle = \int_0^1 e^{2\pi i n x} \cdot \overline{e^{2\pi i n x}} \, dx =$$

$$= \int_0^1 e^{2\pi i n x} \cdot e^{-2\pi i n x} \, dx = \int_0^1 e^{2\pi i n x - 2\pi i n x} \, dx = \int_0^1 e^0 \, dx = \int_0^1 1 \, dx = 1.$$

$n \neq m.$

$$\langle e_n, e_m \rangle = \int_0^1 e^{2\pi i n x} \cdot \overline{e^{2\pi i m x}}\, dx = \int_0^1 e^{2\pi i n x} \cdot e^{-2\pi i m x}\, dx =$$

$$= \int_0^1 e^{2\pi i n x - 2\pi i m x}\, dx = \int_0^1 e^{2\pi i (n-m)x}\, dx =$$

$$= \int_0^1 \Big(\cos\big(2\pi(n-m)x\big) + i \sin\big(2\pi(n-m)x\big) \Big) dx =$$

$$= \int_0^1 \cos\big(2\pi(n-m)x\big)\, dx + i \int_0^1 \sin\big(2\pi(n-m)x\big)\, dx =$$

$$= \left[\frac{1}{2\pi(n-m)} \sin\big(2\pi(n-m)x\big) \right]\Bigg|_0^1 + i \cdot \left[-\frac{1}{2\pi(n-m)} \cos\big(2\pi(n-m)x\big) \right]\Bigg|_0^1 =$$

$$= \frac{\sin\big(2\pi(n-m)\big) - \sin(0)}{2\pi(n-m)} - i \cdot \frac{\cos\big(2\pi(n-m)\big) - \cos(0)}{2\pi(n-m)} =$$

$$= \frac{0-0}{2\pi(n-m)} - i \, \frac{1-1}{2\pi(n-m)} = \underline{0}.$$

Remark. If $f : [0,1] \longrightarrow \mathbb{C}$, $f(x) = f_1(x) + i \cdot f_2(x)$

with $f_1, f_2 : [0,1] \longrightarrow \mathbb{R}$;

$$f_1(x) = \mathrm{Re}\big(f(x)\big) = \frac{1}{2}\left(f(x) + \overline{f(x)} \right)$$

$$f_2(x) = \mathrm{Im}\big(f(x)\big) = \frac{1}{2i}\left(f(x) - \overline{f(x)} \right)$$

and :

$$\int_0^1 f(x)\,dx = \int_0^1 f_1(x)\,dx + i \int_0^1 f_2(x)\,dx.$$

② DENSE APPROXIMATION:

It is based on the Weierstrass' Approximation Theorem:

For any continuous function $f : [0,1] \to \mathbb{C}$ and $\varepsilon > 0$ there is $N \geq 0$ integer and $c_0, c_1 \dots c_N \in \mathbb{C}$ such that

$$\max_{x \in [0,1]} \left| f(x) - (c_0 + c_1 x + \dots + c_N \cdot x^N) \right| \leq \varepsilon.$$

"Trigonometric Polynomial":

$$\left(c_{-N} z^{-N} + c_{-N+1} \cdot z^{-N+1} + \dots + c_0 + c_1 + \dots + c_N \cdot z^N \right) \Big|_{z = e^{2\pi i x}} =$$

$$= \underbrace{c_{-N} e^{-2\pi i N x} + c_{-N+1} e^{-2\pi i (N-1) x} + \dots + c_0 + c_1 e^{2\pi i x} + \dots + c_N e^{2\pi i N x}}_{c_{-N} \cdot e_{-N}(x) + \dots + c_N \cdot e_N(x).}$$

SO WHAT?:

<u>Theorem</u> (Consequence of ONB)[1]. Let $f \in L^2[0,1]$, i.e.

$$f : [0,1] \longrightarrow \mathbb{C}, \quad \int_0^1 |f(x)|^2 dx < \infty$$

Then:

① (Convergence) Let $S_N(x) = c_{-N} \cdot e_{-N}(x) + \cdots + c_{-1} \cdot e_{-1}(x) + c_0 + c_1 \cdot e_1(x) + \cdots + c_N \cdot e_N(x)$

$$= c_{-N} e^{-2\pi i N x} + \cdots + c_{-1} e^{-2\pi i x} + c_0 + c_1 e^{2\pi i x} + \cdots + c_N e^{2\pi i N x}.$$

minus sign because $\overline{e_k(x)} = e^{-2\pi i k x} = e_{-k}(x)$.

where $N \geq 0$ is an integer and:

$$c_k = \langle f, e_k \rangle = \int_0^1 f(x) \cdot e^{-2\pi i k x} dx \in \mathbb{C}.$$

Then:

$$\lim_{N \to \infty} \| f - S_N \| = 0.$$

Explicitly:

$$\lim_{N \to \infty} \int_0^1 | f(x) - c_{-N} e^{-2\pi i N x} - \cdots - c_N e^{2\pi i N x} |^2 dx = 0.$$

[Formally, we write: $\lim_{N \to \infty} S_N = f$, in L^2-sense

(or, in mean-square sense)].

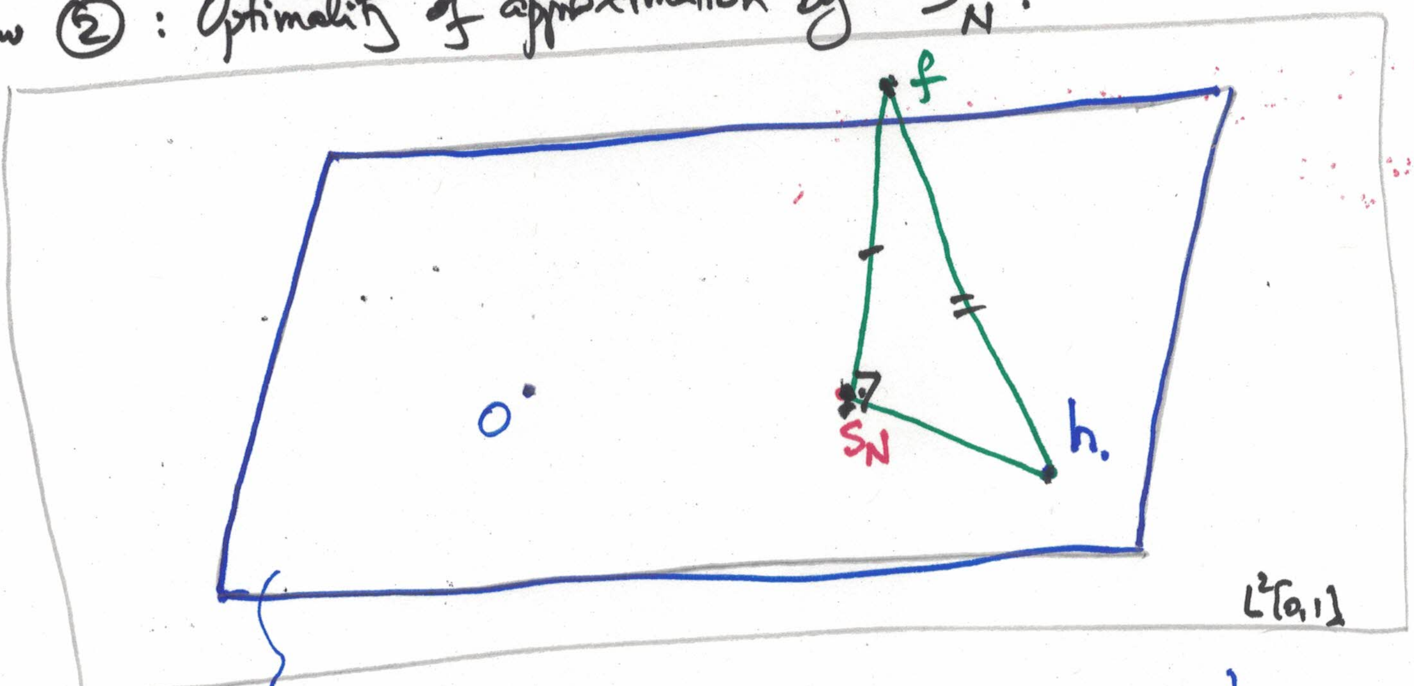
(optimality)

② For any integer $N \geq 0$ and complex numbers $g_{-N}, g_{-N+1}, \dots, g_0, g_1, \dots g_N$

$$\|f - S_N\| \leq \|f - (g_{-N} \cdot e_{-N} + \dots + g_N \cdot e_N)\|.$$

Why:

Show ②: Optimality of approximation by S_N.



$$\mathcal{E}_N = \operatorname{span}\{e_{-N}, e_{-N+1}, \dots, e_0, e_1, \dots e_N\} = \{g_{-N} \cdot e_{-N} + \dots + g_N \cdot e_N : g_{-N}, \dots g_N \in \mathbb{C}\}.$$

Finite dim. linear space: $\dim_{\mathbb{C}} \mathcal{E}_N = 2N+1$.

$$h = g_{-N} e_{-N} + \dots + g_N e_N \in \mathcal{E}_N.$$

Claim: $\|f - S_N\|^2 + \|S_N - h\|^2 = \|f - h\|^2$. (*)

Once shown $\rightarrow$ $\|f-h\|^2 \geq \|f - S_N\|^2 \longrightarrow \|f - (g_{-N} e_{-N} + \dots + g_N e_N)\| \geq \|f - S_N\|$

(what we want).

How to get (*) (claim):

$$\|f-h\|^2 = \langle f-h, f-h\rangle = \underbrace{\langle f,f\rangle}_{\|f\|^2} - \langle h,f\rangle - \langle f,h\rangle + \langle h,h\rangle$$

$$\langle h,f\rangle = \langle g_{-N}e_{-N} + \dots + g_N e_N, f\rangle = g_{-N}\cdot\langle e_{-N}, f\rangle + \dots + g_N\langle e_N, f\rangle =$$

$$= g_{-N}\cdot\overline{c}_{-N} + g_{-N+1}\cdot\overline{c}_{-N+1} + \dots + g_N\cdot\overline{c}_N$$

where $c_k = \langle f, e_k\rangle = \int_0^1 f(x)\, e^{-2\pi i k x}\,dx.$

$$\langle f,h\rangle = \dots = c_{-N}\cdot\overline{g}_{-N} + c_{-N+1}\cdot\overline{g}_{-N+1} + \dots + c_N\overline{g}_N$$

$$\langle h,h\rangle = \langle g_{-N}e_{-N} + \dots + g_N e_N, g_{-N}e_{-N} + \dots + g_N e_N\rangle =$$

$$= g_{-N}\overline{g}_{-N}\overbrace{\langle e_{-N}, e_{-N}\rangle}^{1} + \overset{0}{\dots} + g_N\overline{g}_{-N}\overbrace{\langle e_N, e_{-N}\rangle}^{0} + \dots$$

$$+ g_N\overline{g}_{-N}\underbrace{\langle e_N, e_{-N}\rangle}_{0} + \dots + g_N\overline{g}_N\underbrace{\langle e_N, e_N\rangle}_{1} =$$

$$= |g_{-N}|^2 + |g_{-N+1}|^2 + \dots + |g_N|^2$$

$$\|f-h\|^2 = \|f\|^2 - g_{-N}\overline{c}_{-N} - \dots - g_N\overline{c}_N - c_{-N}\overline{g}_{-N} - \dots - c_N\overline{g}_N +$$

$$+ |g_{-N}|^2 + \dots + |g_N|^2.$$

$$\|f-S_N\|^2 = \|f\|^2 - c_{-N}\bar{c}_{-N} - \dots - c_N\bar{c}_N - \underbrace{\cancel{c_{-N}\bar{c}_{-N}} - \dots - \cancel{c_N\bar{c}_N}}$$

$$+\cancel{|c_{-N}|^2} + \dots + \cancel{|c_N|^2} =$$

$$= \|f\|^2 - |c_{-N}|^2 - \dots - |c_N|^2$$

$$\|f-h\|^2 - \|f-S_N\|^2 = |g_{-N}|^2 + \dots + |g_N|^2 - g_{-N}\bar{c}_{-N} - \dots - g_N\bar{c}_N -$$

$$- c_{-N}\bar{g}_{-N} - \dots - c_N\bar{g}_N + |c_{-N}|^2 + \dots + |c_N|^2 =$$

$$= |g_{-N} - c_{-N}|^2 + \dots + |g_N - c_N|^2$$

check: $\|S_N - h\|^2 = |g_{-N} - c_{-N}|^2 + \dots + |g_N - c_N|^2$

$\longrightarrow$ Shows (*) claim:

$$\boxed{S_N = \operatorname*{argmin}_{h \in \mathcal{E}_N} \|f-h\|}$$

By Dense Approximation property: $\lim_{N\to\infty} \|f - S_N\| = 0$.

Notation: C_K : k^{th} Fourier coefficient of f ; $\sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x}$ is called the Fourier series