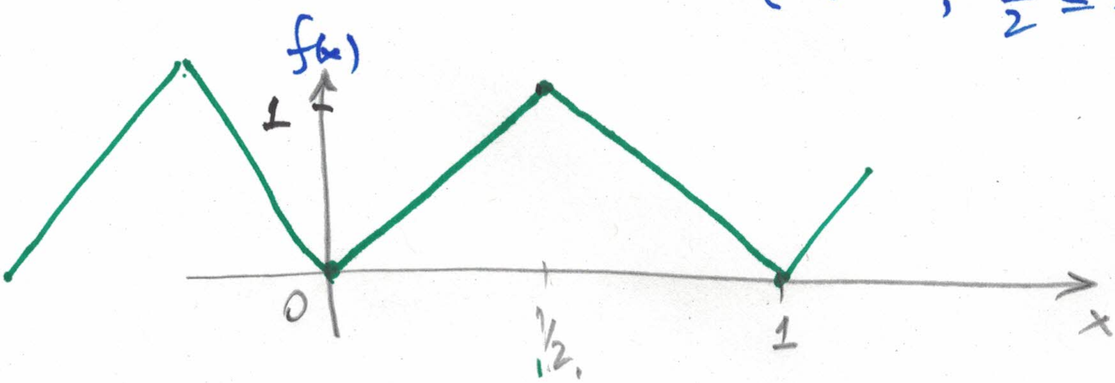
Fourier Series (2)

Example:

$f : [0,1] \longrightarrow \mathbb{R}$, $\quad f(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2-2x, & \frac{1}{2} \le x \le 1 \end{cases}$



Think of f as the restriction of a 1-periodic function $\tilde{f}$ defined on $\mathbb{R}$, $\quad \tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$,

$\tilde{f}$ is the 1-periodic extension of f,

$$\tilde{f}(x) = f(\underbrace{x \text{ modulo } 1}_{x-N \text{ s.t. } \underset{\text{floor}(x)}{\overset{}{N}} \le x < N+1,\ N \in \mathbb{Z} \text{ (integer)}}).$$

N " floor(x)

Want: Compute the Fourier Series expansion of f (equivalently, of $\tilde{f}$)

Step 1: Compute the Fourier coefficients.

$$c_n = \int_0^1 f(x)\, e^{-2\pi i n x}\, dx \quad , \quad n \in \mathbb{Z} \text{ (integer)}.$$

Case 1: $n=0$: $c_0 = \int_0^1 f(x)\,dx = \text{Area}(\Delta) = \frac{1\cdot 1}{2} = \frac{1}{2}$

$e^{2\pi i\cdot 0\cdot x} = 1$

Case 2: $n \neq 0$:

$$c_n = \int_0^1 f(x) e^{-2\pi i n x}\,dx = \underbrace{\int_0^{1/2} 2x\, e^{-2\pi i n x}\,dx}_{I_1} + \underbrace{\int_{1/2}^{1} (2-2x) e^{-2\pi i n x}\,dx}_{I_2} = I_1 + I_2$$

$$I_1 = \int_0^{1/2} 2x\, e^{-2\pi i n x}\,dx \overset{*}{=}$$

$$e^{-2\pi i n x} = \frac{d}{dx}\left(\frac{1}{-2\pi i n}\, e^{-2\pi i n x}\right), \quad \int_a^b u\,dv = u\cdot v\Big|_a^b - \int_a^b v\,du$$

$$\overset{*}{=} \int_0^{1/2} 2x\, d\left(\frac{1}{-2\pi i n} e^{-2\pi i n x}\right) = \left(2x\, \frac{1}{-2\pi i n}\, e^{-2\pi i n x}\right)\Bigg|_0^{1/2} - \int_0^{1/2} \frac{1}{-\cancel{2}\pi i n}\, e^{-2\pi i n x}\cdot \cancel{2}\,dx$$

$$= \left(\frac{1}{-2\pi i n}\, e^{-\pi i n} - 0\right) + \frac{1}{\pi i n}\int_0^{1/2} e^{-2\pi i n x}\,dx =$$

$$= -\frac{1}{2\pi i n} e^{-\pi i n} + \frac{1}{\pi i n}\,\frac{1}{-2\pi i n}\, e^{-2\pi i n x}\Big|_0^{1/2} = -\frac{e^{-\pi i n}}{2\pi i n} + \frac{e^{-\pi i n}}{2\pi^2 n^2} - \frac{1}{2\pi^2 n^2}$$

Remark: When n is an integer:

$$e^{-\pi i n} = \cos(-\pi n) + i\underbrace{\sin(-\pi n)}_{=0 \text{ always.}} = \cos(\pi n) = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases} = (-1)^n$$

$$e^{\pi i n} = \cos(\pi n) + i\underbrace{\sin(\pi n)}_{=0} = \cos(\pi n) = (-1)^n.$$

$$\boxed{e^{\pi i n} = e^{-\pi i n} = (-1)^n = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd.} \end{cases}}$$

$$I_2=\int_{1/2}^{1}(2-2x)\,e^{-2\pi i n x}\,dx = \int_{1/2}^{1}(2-2x)\frac{d}{dx}\left(-\frac{1}{2\pi i n}e^{-2\pi i n x}\right)dx =$$

$$= -\frac{2-2x}{2\pi i n}e^{-2\pi i n x}\Big|_{1/2}^{1} - \frac{2}{2\pi i n}\int_{1/2}^{1} e^{-2\pi i n x}\,dx =$$

$$= -\frac{0-e^{-\pi i n}}{2\pi i n} - \frac{2}{2\pi i n}\,\frac{1}{-2\pi i n}\,e^{-2\pi i n x}\Big|_{1/2}^{1} =$$

$$= \frac{e^{-\pi i n}}{2\pi i n} - \frac{2}{4\pi^2 n^2}\left(e^{-2\pi i n} - e^{-\pi i n}\right)$$

Remark. when n is an integer:

$$e^{-2\pi i n} = \cos(-2\pi n) + i\underbrace{\sin(-2\pi n)}_{0} = \cos(0) = 1.$$

$$e^{2\pi i n} = \cos(2\pi n) + i\underbrace{\sin(2\pi n)}_{=0} = \cos(0) = 1.$$

$$\boxed{e^{2\pi i n} = e^{-2\pi i n} = 1}$$

$$I_2 = \frac{(-1)^n}{2\pi i n} - \frac{1}{2\pi^2 n^2}\left(1-(-1)^n\right).$$

$$I_1 = -\frac{(-1)^n}{2\pi i n} + \frac{(-1)^n - 1}{2\pi^2 n^2}$$

$$c_n = I_1 + I_2 = \frac{(-1)^n - 1}{\pi^2 n^2} = \begin{cases} 0, & n \text{ even.}, n \neq 0 \\ -\dfrac{2}{\pi^2 n^2}, & n \text{ odd.} \end{cases}$$

Thus:
$$c_n = \begin{cases} \frac{1}{2}, & n = 0 \\ 0, & n \text{ even}, n \neq 0 \\ -\frac{2}{\pi^2 n^2}, & n \text{ odd}. \end{cases}$$

(this) Fourier Coefficients of f.

Step 2 Assemble the Fourier Series.

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x} = \frac{1}{2} + \sum_{\substack{n\neq 0 \\ n\in\mathbb{Z}}} c_n e^{2\pi i n x} = \frac{1}{2} + \sum_{\substack{n\in\mathbb{Z} \\ n \text{ odd}}} \frac{-2}{\pi^2 n^2} e^{2\pi i n x} =$$

n odd : $n = 2p+1$.

$$= \frac{1}{2} - \frac{2}{\pi^2} \sum_{p=-\infty}^{\infty} \frac{1}{(2p+1)^2} e^{2\pi i (2p+1) x}$$

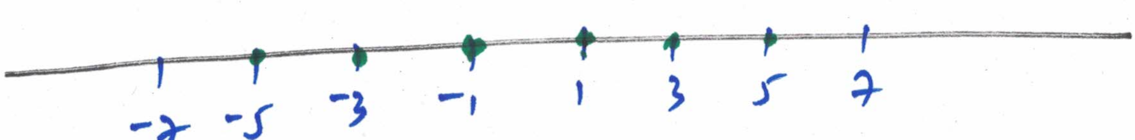
Fourier Series of f

But: $f(x)$ is real $\longrightarrow$ $f(x) = \text{Real}(f(x))$.

$$\text{Real}\left(\frac{1}{2} - \frac{2}{\pi^2} \sum_{p=-\infty}^{\infty} \frac{1}{(2p+1)^2} e^{2\pi i (2p+1)x} \right) =$$

$$= \frac{1}{2} - \frac{2}{\pi^2} \sum_{p=-\infty}^{\infty} \frac{1}{(2p+1)^2} \underbrace{\text{Real}\left(e^{2\pi i (2p+1)x} \right)}_{\cos(2\pi(2p+1)x)} =$$

$$= \frac{1}{2} - \frac{2}{\pi^2} \sum_{p=-\infty}^{\infty} \frac{1}{(2p+1)^2} \cos\left(2\pi(2p+1)x\right) = \underbrace{\frac{1}{2} - \frac{4}{\pi^2} \sum_{p=0}^{\infty} \frac{\cos(2\pi(2p+1)x)}{(2p+1)^2}}_{\text{Fourier Series.}}$$



"Sanity Checks":

1) $\tilde{f}$ is even $\Longrightarrow$ Fourier Series involves cosine functions only.

2) $\tilde{f}$ is symmetric about $x=\frac{1}{2}$ $\Rightarrow$ $1/2$ of all integers are involved in the Series.

3) $\tilde{f}$ is continuous but not differentiable, i.e. $\tilde{f} \in C^0$ (but not C^1).

$$\longrightarrow \quad c_n \sim \frac{1}{n^2}$$

For this function:

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{p=0}^{\infty} \frac{\cos(2\pi(2p+1)x)}{(2p+1)^2}$$

Let's check/observe using Matlab.

Sine – Cosine Series

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x} = c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n \cdot \left(\cos(2\pi n x) + i \sin(2\pi n x)\right) =$$

$$= c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n \cdot \cos(2\pi n x) + i \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n \cdot \sin(2\pi n x) =$$

$$= c_0 + \sum_{n=1}^{\infty}\left(c_n \cdot \cos(2\pi n x) + c_{-n} \cdot \cos(-2\pi n x)\right) +$$

$$+ i \sum_{n=1}^{\infty}\left(c_n \cdot \sin(2\pi n x) + c_{-n} \cdot \sin(-2\pi n x)\right) =$$

$$= c_0 + \sum_{n=1}^{\infty}(c_n + c_{-n}) \cos(2\pi n x) + i \sum_{n=1}^{\infty}(c_n - c_{-n}) \sin(2\pi n x).$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos(2\pi n x) + \sum_{n=1}^{\infty} b_n \cdot \sin(2\pi n x).$$

$$a_n = c_n + c_{-n} = \int_0^1 e^{-2\pi i n x} f(x)\,dx + \int_0^1 e^{2\pi i n x} f(x)\,dx =$$

$$= \int_0^1 \underbrace{\left(e^{2\pi i n x} + e^{-2\pi i n x}\right)}_{2\cos(2\pi n x)} f(x)\,dx = 2\int_0^1 f(x) \cos(2\pi n x)\,dx$$

$$b_n = i(c_n - c_{-n}) = i\int_0^1 e^{-2\pi i n x} f(x)\,dx - i\int_0^1 e^{2\pi i n x} f(x)\,dx = i\int_0^1 \underbrace{\left(e^{-2\pi i n x} - e^{2\pi i n x}\right)}_{-2i\sin(2\pi n x)} f(x)\,dx$$

$$b_n = i\cdot(-2i)\int_0^1 f(x)\sin(2\pi n x)\,dx = 2\int_0^1 f(x)\sin(2\pi n x)\,dx.$$

$$a_0 = c_0 = \int_0^1 f(x)\,dx.$$

To summarize:

Sine - Cosine Series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(2\pi n x) + \sum_{n=1}^{\infty} b_n \sin(2\pi n x)$$

where: $a_0 = c_0 = \int_0^1 f(x)\,dx$

$$a_n = 2\int_0^1 f(x)\cos(2\pi n x)\,dx, \quad b_n = 2\int_0^1 f(x)\sin(2\pi n x)\,dx$$

Fourier Series for T - periodic functions, or $f:[a,b]\longrightarrow\mathbb{C}$,

where $\underline{b-a=T}$.

<u>Idea</u>: $e_n(x) = e^{2\pi i n x} \dashrightarrow \varphi_n(x) = \frac{1}{\sqrt{T}}e^{\frac{2\pi i n x}{T}}$

<u>Fact</u>: $\{\varphi_n,\ n\in\mathbb{Z}\}$ is ONB for $\underbrace{L^2[a,b]}_{\left\{f:[a,b]\to\mathbb{C}\ \middle|\ \int_a^b |f(x)|^2dx<\infty\right\}}$.

Then:

$$f = \sum_{n=-\infty}^{\infty} \langle f, \varphi_n \rangle \varphi_n$$

$$f(x) = \sum_{n=-\infty}^{\infty} F[n] \cdot e^{\frac{2\pi i n x}{T}}$$

where: $$\underbrace{\boxed{F[n] = \frac{1}{b-a} \int_a^b e^{-\frac{2\pi i n x}{T}} f(x)\,dx}}_{n^{th} \text{ Fourier Coff. of } f \text{ on } [a,b].}$$

$$T = b-a$$

Similarly.

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi n x}{T}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi n x}{T}\right)$$

where: $A_0 = F[0] = \frac{1}{T} \int_a^b f(x)\,dx$ --→ average of f

$$A_n = \frac{2}{T} \int_a^b f(x) \cos\left(\frac{2\pi n x}{T}\right) dx$$

$$B_n = \frac{2}{T} \int_a^b f(x) \sin\left(\frac{2\pi n x}{T}\right) dx$$

Remark: ONB : $\left\{ \frac{1}{\sqrt{T}}, \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi n x}{T}\right), \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi n x}{T}\right) ; n \geq 1 \right\}$.