# Convergence Results For Fourier Series

**Recap:** $L^2$ - convergence / Mean-Square Convergence.

**Setup:** $f : [0,1) \longrightarrow \mathbb{C}$ s.t. $\int_0^1 |f(x)|^2 dx < \infty$.

We showed: $\{e_n(x) = e^{2\pi i n x}, n \in \mathbb{Z}\}$ is ONB for $L^2[0,1]$.

**Consequences:**

① $\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x} = f(x)$, in $L^2$-sense.

where $c_n = \int_0^1 e^{-2\pi i n x} f(x) dx$.

$$\lim_{N\to\infty} \int_0^1 \left| f(x) - \sum_{n=-N}^{N} c_n e^{2\pi i n x} \right|^2 dx = 0$$

($L^2$-convergence or, mean-square convergence)

② $$\left[ \underbrace{\int_0^1 |f(x)|^2 dx}_{\|f\|^2} = \langle f, f \rangle \underset{\uparrow}{=} \sum_{n=-\infty}^{\infty} |c_n|^2 \right]$$ Parseval - Plancherel Identity.

because $\{e_n, n \in \mathbb{Z}\}$ ONB.

$\|f\|^2 = \|c\|_2^2$

or: $\|f\| = \|c\|_2$

"Signal Energy is preserved in coefficients energy".

$$\xrightarrow{f} \underbrace{\boxed{\langle \cdot , e_n \rangle}}_{\substack{\text{Analysis} \\ \text{(Decomposition)}}} \xrightarrow{(c_n)_n} \underbrace{\boxed{\sum_{n=-\infty}^{\infty} c_n e_n}}_{\substack{\text{Synthesis.} \\ \text{(Reconstruction).}}} \xrightarrow{f}$$

Unitary map.

$$\|f\| = \|c\|$$

③ ② $\Rightarrow$ By "polarization":

If. $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$, $c_n = \int_0^1 e^{-2\pi i n x} f(x)\,dx$

$g(x) = \sum_{n=-\infty}^{\infty} d_n e^{2\pi i n x}$, $d_n = \int_0^1 e^{-2\pi i n x} g(x)\,dx$

Then: complex conjugate. complex conjugate.

$$\boxed{\int_0^1 f(x) \cdot \overline{g(x)}\,dx = \langle f, g \rangle = \langle c, d \rangle_2 = \sum_{n=-\infty}^{\infty} c_n \overline{d_n}}$$

Parseval - - Plancherel Identity.

what about convergence of $\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$ at a fixed $x$ ?

<u>Theorem</u> [Dirichlet ~~Criterion~~ Convergence Result]

Assume $f : [0,1) \longrightarrow \mathbb{C}$ satisfies "some" regularity conditions,

Then, for every $x \in [0,1)$,

$$\lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{2\pi i n x} = \frac{1}{2}\left(f(x-0) + f(x+0)\right)$$

where: $c_n = \int_0^1 e^{-2\pi i n x} f(x)\,dx$ ($n^{th}$ Fourier coefficient)

$f(x-0) = \lim_{y \uparrow x} f(y) = \lim_{\substack{y\to x \\ y<x}} f(y)$ (left-side limit)

$f(x+0) = \lim_{y \downarrow x} f(y) = \lim_{\substack{y\to x \\ y>x}} f(y)$ (right-side limit).

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"Some" regularity conditions:

1) $f$ is bounded: $\exists M_0 > 0$ s.t. $|f(x)| \leq M_0$, for every $x$.

2) There are $t_0 = 0 \leq t_1 < t_2 < \dots < t_L \leq 1 = t_{L+1}$ such that:

For each $0 \leq k \leq L$ → i) $f\big|_{(t_k, t_{k+1})}$ is of class $C^1$ (or more). (continuous, differentiable with continuous $1^{st}$ derivative)

→ ii) At every $t_k$, $\lim_{x \uparrow t_k} f'(x)$, $\lim_{x \downarrow t_k} f'(x)$ exist and are finite.

Remarks:

1) In Dirichlet's Theorem, $f$ does not have to be continuous.

2) (Fact) It is not true. that the Fourier series of a continuous function to converge pointwise Everywhere. (for every $x$

"Pointwise Convergence" : convergence at a specific $x$

"Pointwise Convergence Everywhere" : convergence at every $x$.

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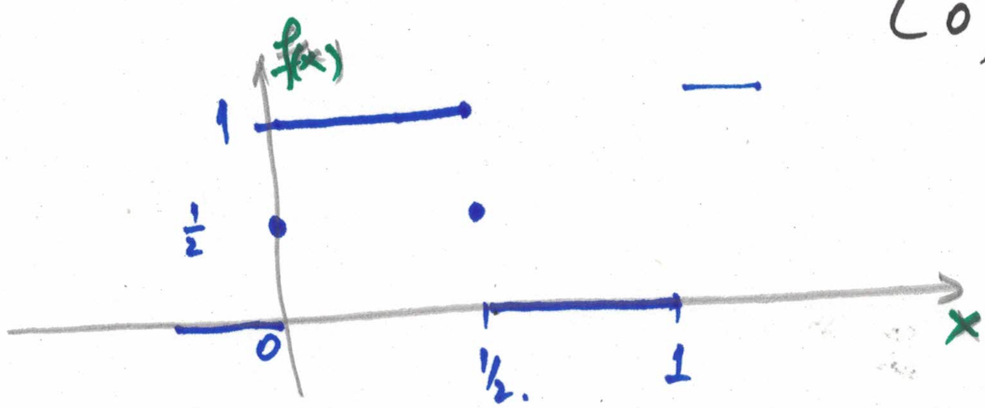
Example.

$f: [0,1) \longrightarrow \mathbb{R}$, $\qquad f(x) = \begin{cases} 1, & 0 < x < \frac{1}{2} \\ \frac{1}{2}, & x = \frac{1}{2}, \text{ or } x = 0. \\ 0, & \frac{1}{2} < x < 1. \end{cases}$




Want: 1) Expand $f$ in Fourier Series.

2) Analyze pointwise convergence. of the Fourier Series.

Fourier Coefficients: $c_n = \int_0^1 e^{-2\pi i n x} f(x)\,dx$

$n=0 \longrightarrow c_0 = \int_0^1 f(x)dx = \frac{1}{2}$

$n \neq 0$

$$c_n = \int_0^1 e^{-2\pi i n x} f(x)\,dx = \int_0^{1/2} e^{-2\pi i n x}\,dx =$$

$$= \frac{1}{-2\pi i n} e^{-2\pi i n x} \Big|_0^{1/2} = \frac{e^{-i\pi n} - 1}{-2\pi i n} = \frac{(-1)^n - 1}{-2\pi i n}$$

$$c_n = \begin{cases} 0, & n \text{ even}, n \neq 0 \\ \frac{1}{\pi i n}, & n \text{ odd}. \\ \frac{1}{2}, & n = 0. \end{cases} \quad , n \in \mathbb{Z}$$

Fourier Series:

$$n \neq 0$$

$$a_0 = c_0 = \frac{1}{2}, \quad a_n = c_n + c_{-n} = \begin{cases} 0, & n \text{ even}. \\ \frac{1}{\pi i n} + \frac{1}{-\pi i n} = 0, & n \text{ odd} \end{cases}$$

$$b_n = i \cdot (c_n - c_{-n}) = \begin{cases} 0, & n \text{ even}. \\ i\left(\frac{1}{\pi i n} - \frac{1}{-\pi i n}\right) = \frac{2}{\pi n}, & n \text{ odd}. \end{cases}$$

We obtained:

$$a_0 = \frac{1}{2}, \quad a_n = 0 \text{ for } n \geq 1 \;; \quad b_n = \begin{cases} 0, & n \text{ even} \\ \frac{2}{\pi n}, & n \text{ odd}. \end{cases}$$

$$n = 2p+1$$

Fourier Series:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n x) + \sum_{n=1}^{\infty} b_n \sin(2\pi n x) = \frac{1}{2} + \sum_{p=0}^{\infty} \frac{2}{\pi(2p+1)} \sin(2\pi(2p+1)x)$$

$$= \frac{1}{2} + \frac{2}{\pi} \sum_{p=0}^{\infty} \frac{\sin(2\pi(2p+1)x)}{2p+1}$$

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{p=0}^{\infty} \frac{\sin(2\pi(2p+1)x)}{2p+1}.$$

<u>Pointwise Convergence:</u>

by Dirichlet's Th.

At every $x$, $\frac{1}{2} + \frac{2}{\pi}\lim_{N\to\infty}\sum_{p=0}^{N} \frac{\sin(2\pi(2p+1)x)}{2p+1} = \frac{1}{2}(f(x-0)+f(x+0))$

If $f$ is continuous at $x$, then. $\frac{1}{2}(f(x-0)+f(x+0)) = f(x)$.

If $f$ is discontinuous then we need to compute the average of side limi[ts]

At $x=0$:
$f(0-0)=0$, $f(0+0)=1 \Rightarrow \frac{1}{2}(f(0-0)+f(0+0)) = \frac{1}{2} = f(0)$

Matches our definition of $f$ at $0$ and $1/2$

At $x=\frac{1}{2}$, $f(\frac{1}{2}-0)=1$, $f(\frac{1}{2}+0)=0$

$\Rightarrow \frac{1}{2}(f(\frac{1}{2}-0)+f(\frac{1}{2}+0)) = \frac{1}{2} = f(\frac{1}{2})$

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Therefore: The Fourier Series converges pointwise at every $x$ (everywhere) to the function $f(x)$.

for every $0 \le x < 1$,

$$\Rightarrow \frac{1}{2} + \frac{2}{\pi}\lim_{N\to\infty}\sum_{p=0}^{N} \frac{\sin(2\pi(2p+1)x)}{2p+1} = \begin{cases} 1, & 0<x<\frac{1}{2} \\ \frac{1}{2}, & \text{for } x=0 \text{ or } x=\frac{1}{2} \\ 0, & \text{for } \frac{1}{2}<x<1. \end{cases}$$

At $x=0$:

$$\text{Fourier Series} = \frac{1}{2} + \frac{2}{\pi} \lim_{N\to\infty} \sum_{p=0}^{N} \frac{\sin(0)}{2p+1} = \underline{\underline{\frac{1}{2}}}$$

$$f(0) = \underline{\underline{\frac{1}{2}}}$$

Similarly: $x = \frac{1}{2} \longrightarrow \frac{1}{2} + \frac{2}{\pi} \lim_{N\to\infty} \sum_{p=0}^{N} \frac{\overbrace{\sin(\pi(2p+1))}^{=0}}{2p+1} = \frac{1}{2}$

At $x = \frac{1}{4}$:

$$\text{Fourier Series}: \frac{1}{2} + \frac{2}{\pi} \sum_{p=0}^{\infty} \frac{\sin\left(\frac{\pi}{2}(2p+1)\right)}{2p+1} =$$

$$= \frac{1}{2} + \frac{2}{\pi} \sum_{p=0}^{\infty} \frac{\sin\left(\frac{\pi}{2} + p\pi\right)}{2p+1} =$$

$$\sin\left(\frac{\pi}{2} + p\cdot\pi\right) = \begin{cases} 1, & p = 0,2,4,\dots \quad (\text{even}) \\ -1, & p = 1,3,5,\dots \quad (\text{odd}) \end{cases}$$

$$= \frac{1}{2} + \frac{2}{\pi} \underbrace{\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \dots + \frac{(-1)^p}{2p+1}\right)}_{\text{alternating sum}} \Rightarrow$$

$$f\left(\tfrac{1}{4}\right) = 1.$$

$$\Rightarrow 1 = \frac{1}{2} + \frac{2}{\pi}(\dots) \Rightarrow \boxed{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}.}$$

<u>Notice</u>: $\sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1}$ is not <u>absolutely convergent</u>:

$$\sum_{p=0}^{\infty} \left| \frac{(-1)^p}{2p+1} \right| = \sum_{p=0}^{\infty} \frac{1}{2p+1} = \frac{1}{2} \sum_{p=0}^{\infty} \frac{1}{p+\frac{1}{2}} \sim \int_1^{\infty} \frac{dx}{x} = \infty$$

Application of Parseval – Plancherel Identity:

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \int_0^1 |f(x)|^2 dx$$

$$\frac{1}{4} + 2\frac{1}{\pi^2}\sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} = \frac{1}{2}$$

$$\sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} = \frac{1}{4}\cdot\frac{\pi^2}{2} = \frac{\pi^2}{8}$$

$$\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}\right].$$

Want: $S = \sum_{n=1}^{\infty} \frac{1}{n^2} = ?$

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \underbrace{1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}_{\text{odd terms}} +$$

$$+ \sum_{p=1}^{\infty} \frac{1}{(2p)^2} = \frac{\pi^2}{8} + \frac{1}{4}\sum_{p=1}^{\infty}\frac{1}{p^2} = \frac{\pi^2}{8} + \frac{1}{4}\cdot S$$

$$\Rightarrow \frac{3}{4}S = \frac{\pi^2}{8} \Rightarrow S = \frac{\pi^2}{6}.$$

$$\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}\right].$$