L7

Truncation / Approximation Errors

$f:[0,1] \longrightarrow \mathbb{C}$, $\{e_n(x) = e^{2\pi i n x}, n \in \mathbb{Z}\}$ ONB for $L^2[0,1]$.

Fourier Series: $f \to \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$, $c_n = \int_0^1 e^{-2\pi i n x} f(x)\,dx$.

Convergence:

L^2- or Mean Square Convergence:

$$\lim_{N\to\infty} \int_0^1 |f(x) - A_N(x)|^2 dx = 0$$

Pointwise Convergence : Assuming f satisfies "some" regularity conditions:

$\forall x \in [0,1]$, $\lim_{N\to\infty} A_N(x) = \frac{1}{2}\left(f(x-0) + f(x+0)\right)$

where : $A_N(x) = \sum_{k=-N}^{N} c_k e^{2\pi i k x}$ $\longrightarrow$ N^{th} partial Fourier Sum.

Problem : What can we say about:

$$\|f - A_N\|^2 = \int_0^1 |f(x) - A_N(x)|^2 dx$$

or.

$$\max_{x\in[0,1]} |f(x) - A_N(x)| =: \|f - A_N\|_\infty .$$

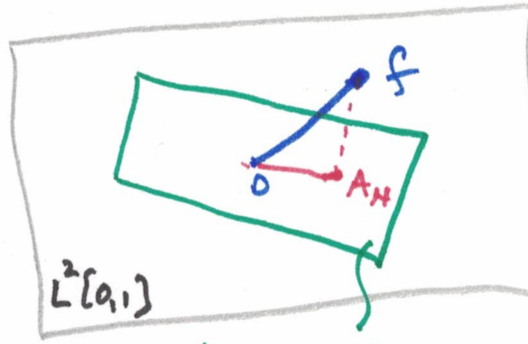
Mean Square Error (MSE):

$$MSE = \int_0^1 \Big| f(x) - \sum_{k=-N}^{N} c_k e^{2\pi i k x} \Big|^2 dx.$$

Remark:

$\{ e_n, n \in \mathbb{Z} \}$ ONB $\longrightarrow$

$$\|f\|^2 = \|A_N\|^2 + \|f - A_N\|^2$$



$\mathcal{E}_N = \text{Span}\{ e_{-N}, \dots, e_N \}.$

2N+1 dimensional space.

$$\Rightarrow \|f - A_N\|^2 = \|f\|^2 - \|A_N\|^2 =$$

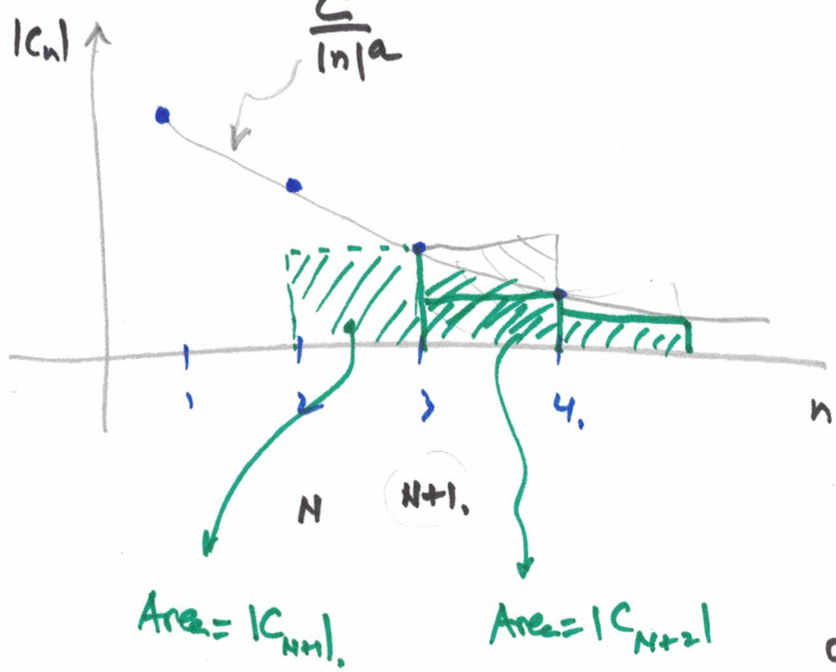
$$= \sum_{k=-\infty}^{\infty} |c_k|^2 - \sum_{k=-N}^{N} |c_k|^2 = \sum_{k=-\infty}^{-N-1} |c_k|^2 + \sum_{k=N+1}^{\infty} |c_k|^2$$

$$\boxed{MSE = \sum_{k=-\infty}^{-N-1} |c_k|^2 + \sum_{k=N+1}^{\infty} |c_k|^2}$$

Formula: Assume: $|c_n| \leq \dfrac{C}{|n|^a}$, for $|n| \geq 1$.

Question: $\underbrace{\sum_{n=N+1}^{\infty} |c_n| = ?}$ or $\underbrace{\sum_{n=N+1}^{\infty} |c_n|^2 = ?}$

How to estimate these sums?



$$\sum_{n=N+1}^{\infty} \frac{C}{|n|^a} = \sum \text{"Hashed" Areas} \leq \int_{N}^{\infty} \frac{C}{x^a}\, dx = $$

$$= \frac{C}{(a-1)\, N^{a-1}}$$

How good this bound?

$$\sum_{n=N+1}^{\infty} \frac{C}{|n|^a} \geq \int_{N+1}^{\infty} \frac{C}{x^a}\, dx = \frac{C}{(a-1)(N+1)^{a-1}}$$

$$\boxed{\frac{C}{(a-1)(N+1)^{a-1}} \leq \sum_{n=N+1}^{\infty} \frac{C}{|n|^a} \leq \frac{C}{(a-1)\, N^{a-1}}}$$

For our problems, we approximate:

$$\sum_{n=N+1}^{\infty} \frac{C}{n^a} \approx \frac{C}{(a-1)\, N^{a-1}}$$

<u>Example</u>. Assume $f : [0,1] \to \mathbb{R}$ has Fourier coefficients that satisfy $|c_n| = \frac{1}{n^2}$, $|n| \geq 1$. $(n \neq 0)$.

What should be N such that the MSE $\leq 10^{-4}$.

Solution:

$$\text{MSE} = \|f - A_N\|^2 = \sum_{k=-\infty}^{-N-1} |c_k|^2 + \sum_{k=N+1}^{\infty} |c_k|^2 = 2 \sum_{n=N+1}^{\infty} \frac{1}{n^4} \simeq$$

$$\simeq 2 \cdot \frac{1}{3N^3} \leq 10^{-4}$$

$a = 4$

$C = 1$

want.

$$N^3 \geq \frac{20{,}000}{3} = 6{,}666$$

$$\Rightarrow N \geq \sqrt[3]{6{,}666} = 18.82 \Rightarrow \boxed{N = 19}$$

Pointwise Truncation Error:

Assume f is continuous & satisfies all necessary regularity conditions s.t.

$$\lim_{N\to\infty} |f(x) - A_N(x)| = 0, \quad \text{for every } x.$$

Since. $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x}$, $A_N(x) = \sum_{k=-N}^{N} c_k e^{2\pi i k x}$

$$|f(x) - A_N(x)| = \left| \sum_{k=-\infty}^{-N-1} c_k e^{2\pi i k x} + \sum_{k=N+1}^{\infty} c_k e^{2\pi i k x} \right| \leq$$

Triangle Inequality

$$\leq \left| \sum_{k=-\infty}^{-N-1} c_k e^{2\pi i k x} \right| + \left| \sum_{k=N+1}^{\infty} c_k e^{2\pi i k x} \right| \leq$$

$$\leq \sum_{k=-\infty}^{-N-1} |c_k| \cdot |e^{2\pi i k x}| + \sum_{k=N+1}^{\infty} |c_k| \cdot |e^{2\pi i k x}| =$$

$$|e^{2\pi i \alpha}| = |\cos(2\pi\alpha) + i \sin(2\pi\alpha)| = \sqrt{\cos^2(2\pi\alpha) + \sin^2(2\pi\alpha)}$$

$$\boxed{|e^{2\pi i \alpha}| = 1}$$

$$= \sum_{k=-\infty}^{-N-1} |c_k| + \sum_{k=N+1}^{\infty} |c_k|$$

Summarize:

$$\boxed{|f(x) - A_N(x)| \leq \sum_{k=-\infty}^{-N-1} |c_k| + \sum_{k=N+1}^{\infty} |c_k|, \quad \text{for every } x}$$

Pointwise Approx. Error

or:

$$\boxed{\|f - A_N\|_\infty = \max_{x\in[0,1]} |f(x) - A_N(x)| \leq \sum_{k=-\infty}^{-N-1} |c_k| + \sum_{k=N+1}^{\infty} |c_k|}$$

~~[illegible]~~
(Worst Case Pointwise Approximation Error)
(Worst Case Pointwise Truncation Error)

<u>Example</u> Assume $f:[0,1]\longrightarrow\mathbb{R}$ has Fourier coefficients,

$$|c_n| = \frac{1}{|n|^2}, \; \underline{n \neq 0}. \qquad |c_n| = \frac{1}{n^2} = \frac{1}{|n|^2}$$

Question: $N = ?$ s.t. the pointwise approximation error to be less that 10^{-4} ~~at~~ every x.

<u>Solution:</u>

Need: $\quad |f(x) - A_N(x)| \leq 10^{-4}$, for every x.

We obtain this guarantee by making: $\displaystyle\sum_{k=-\infty}^{-N-1}|c_k| + \sum_{k=N+1}^{\infty}|c_k| \leq 10^{-4}$

$$\text{Sum} = 2\cdot\sum_{n=N+1}^{\infty}\frac{1}{n^2} \underset{\substack{\uparrow\\ a=2\\ C=1}}{\leq} 2\cdot\frac{1}{1\cdot N^1} = \frac{2}{N} \underset{\substack{\uparrow\\ \text{want}}}{\leq} 10^{-4}$$

$N \geq 2\cdot 10^4 = 20{,}000$. Can choose $\underline{N = 20{,}000}$.

For the case: $f:[a,b]\to\mathbb{C}$

$$T=b-a.$$

Fourier Series:

$$f(x)=\sum_{n=-\infty}^{\infty}F[n]e^{\frac{2\pi i n x}{T}} = A_0+\sum_{n=1}^{\infty}A_n\cos\left(\frac{2\pi n x}{T}\right)+\sum_{n=1}^{\infty}B_n\sin\left(\frac{2\pi n x}{T}\right)$$

$$F[n]=\frac{1}{T}\int_a^b e^{-\frac{2\pi i n x}{T}}f(x)\,dx., \quad A_0=F[0]=\frac{1}{T}\int_a^b f(x)\,dx.$$

$$A_n=\frac{2}{T}\int_a^b f(x)\cos\left(\frac{2\pi n x}{T}\right)dx$$

$$B_n=\frac{2}{T}\int_a^b f(x)\sin\left(\frac{2\pi n x}{T}\right)dx.$$

Pointwise Approximation Error:

$$\left|f(x)-\sum_{k=-N}^{N}F[k]e^{\frac{2\pi i k x}{T}}\right| \le \sum_{n=-\infty}^{-N-1}|F[n]|+\sum_{n=N+1}^{\infty}|F[n]|$$

$$\left|f(x)-\left(A_0+\sum_{k=1}^{N}A_k\cos\left(\frac{2\pi k x}{T}\right)+\sum_{k=1}^{N}B_k\sin\left(\frac{2\pi k x}{T}\right)\right)\right| \le \sum_{k=N+1}^{\infty}|A_k|+\sum_{k=N+1}^{\infty}|B_k|$$

Mean Square Error:

$$\int_a^b\left|f(x)-\sum_{k=-N}^{N}F[k]e^{\frac{2\pi i k x}{T}}\right|^2dx = T\cdot\left[\sum_{k=-\infty}^{-N-1}|F[k]|^2+\sum_{k=N+1}^{\infty}|F[k]|^2\right].$$

$$\int_a^b\left|f(x)-\left(A_0+\sum_{k=1}^{N}A_k\cos\left(\frac{2\pi k x}{T}\right)+\sum_{k=1}^{N}B_k\sin\left(\frac{2\pi k x}{T}\right)\right)\right|^2dx=\frac{T}{2}\cdot\left[\sum_{k=N+1}^{\infty}|A_k|^2+\sum_{k=N+1}^{\infty}|B_k|^2\right]$$

$\{f_n\}_{n\in\mathbb{Z}}$ ONB $\longrightarrow$

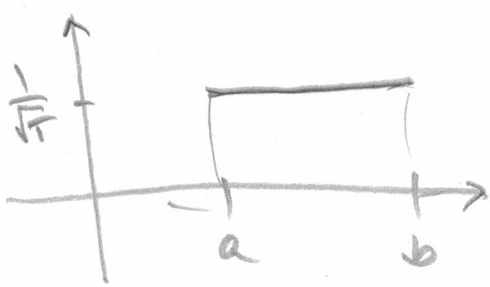
$$f = \sum_{n=-\infty}^{\infty} \langle f, f_n \rangle f_n.$$

$$\left\| f - \sum_{k=-N}^{N} \langle f, f_k \rangle f_k \right\|^2 = \sum_{n=-\infty}^{-N-1} |\langle f, f_n \rangle|^2 + \sum_{n=N+1}^{\infty} |\langle f, f_n \rangle|^2$$

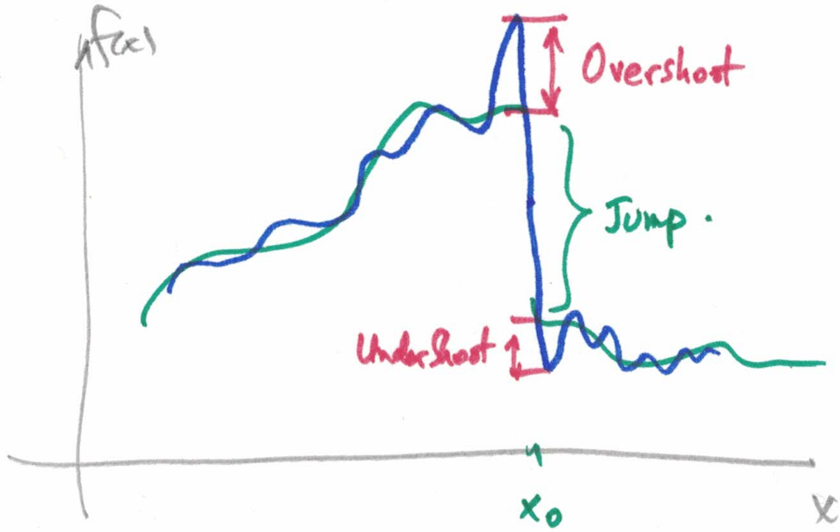
$$\text{ONB: } \left\{ \frac{1}{\sqrt{T}} e^{\frac{2\pi i n x}{T}},\ n \in \mathbb{Z} \right\} \longrightarrow \langle f, f_n \rangle = \frac{1}{\sqrt{T}} \int_a^b e^{-\frac{2\pi i n x}{T}} f(x)\, dx$$

$$= \sqrt{T} \cdot F[n].$$

$$\text{ONB: } \left\{ \frac{1}{\sqrt{T}} \mathbb{1}_{[a,b]},\ \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi n x}{T}\right),\ \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi n x}{T}\right) \right\}_{n \geq 1}.$$



Gibbs Phenomenon:



$\left.\begin{array}{l}|\text{Overshoot}|\\|\text{Undershoot}|\end{array}\right.$ = 9% of the Jump.

$$\text{Jump} = |f(x_0+0) - f(x_0-0)|$$

$$\max_x |f(x) - A_N(x)| = (0.09) \cdot |f(x_0+0) - f(x_0-0)|$$

$$f(x_0+0) = \lim_{\substack{x \to x_0 \\ x > x_0}} f(x).$$

$$f(x_0-0) = \lim_{\substack{x \to x_0 \\ x < x_0}} f(x)$$