I.

1. The solution is

$$u(x,t) = \frac{1}{2}sin(x-t) + \frac{1}{2}sin(x+t) + \frac{1}{2}\int_{x-t}^{x+t}(-cos(y))dy = sin(x-t)$$

2. The plots of $u(x,0) = sin(x)$ and $u(x,\pi) = sin(x-\pi) = -sin(x)$.

II.

3. The solution is

$$u(x,t) = \frac{1}{2}e^{-(x-2t)^2/2} + \frac{1}{2}e^{-(x+2t)^2/2} + \frac{1}{4}\int_{x-2t}^{x+2t} 2ye^{-y^2/2}dy = e^{-(x-2t)^2/2}$$

4. Plot $u(x,0) = e^{-x^2/2}$ and $u(x,1) = e^{-(x-2)^2/2}$.

5. The heat equation has solution

$$u(x,t) = \frac{1}{\sqrt{16\pi t}}\int_{-\infty}^{\infty} e^{-(x-y)^2/16t}e^{-y^2/2}dy = \frac{1}{\sqrt{1+8t}}e^{-\frac{x^2}{2(1+8t)}}$$

The integral can be evaluated using the Fourier transform:

$$f_1(x) = e^{-x^2/16t} \mapsto F_1(s) = \sqrt{16\pi t}e^{-16\pi^2 ts^2}$$

$$f_2(x) = e^{-x^2/2} \mapsto F_2(s) = \sqrt{2\pi}e^{-2\pi^2 s^2}$$

Hence

$$F(s) = F_1(s)F_2(s) = \sqrt{2\pi}\sqrt{16\pi t}e^{-2\pi^2(1+8t)s^2} \mapsto f_1 * f_2(x) = \frac{\sqrt{2\pi}\sqrt{16\pi t}}{\sqrt{2\pi(1+8t)}}e^{-\frac{x^2}{2(1+8t)}}$$

and $u(x,t) = \frac{1}{\sqrt{16\pi t}}f_1 * f_2(x)$.

III

6. The solution is given by

$$u(x,t) = \frac{1}{2}\Lambda((x-3t)\,mod\,12\ -\ 6) + \frac{1}{2}\Lambda((x+3t)\,mod\,12\ -\ 6)$$

where $a\,mod\,12 = a - 12\lfloor\frac{a}{12}\rfloor$ and $\lfloor y\rfloor$ is the largest integer smaller than or equal to y.

7. $u(x,t+T) = u(x,t)$ happens for $3T\,mod\,12 = 0$ that is for $T = 4k$ with k integer. The smallest positive such T is $T = 4$.

8. Plot of $u(x,0) = u(x,4) = \Lambda(x-6)$ and $u(x,1) = \frac{1}{2}\Lambda(x-3) + \frac{1}{2}\Lambda(x-9)$.

IV

9. The Fourier series of the initial condition is

$$u(x,0) = sin(6\pi x) = \frac{1}{2i}e^{6\pi ix} - \frac{1}{2i}e^{-6\pi ix}$$

The solution of the heat equation is

$$u(x,t) = e^{-4\pi^2 9\cdot 9t}\left(\frac{1}{2i}e^{6\pi ix} - \frac{1}{2i}e^{-6\pi ix}\right) = e^{-324\pi^2 t}sin(6\pi x)$$

V.

10.

The sine-series of the initial condition $u(x,0) = sin(6\pi x)$ has one non-zero term for $n = 6$. Hence the non-zero coefficient a_k's b_k's is $a_6 = 1$. Thus

$$u(x,t) = cos(18\pi t)sin(6\pi x)$$