I.

1. Use the Fourier transform of $g(x) = e^{-|x|}$, $G(s) = \frac{2}{1+4\pi^2 s^2}$ and the affine change of coordinate $y = 2x + 5$,

$$F(s) = \int e^{-2\pi isx} e^{-|2x+5|} dx = \frac{1}{2} \int e^{-2\pi is(y-5)/2} e^{-|y|} dy = \frac{e^{5\pi is}}{1 + \pi^2 s^2}$$

2. Use the dilation rule $y = \sqrt{\frac{3}{\pi}} x$ for the Fourier transform $\gamma(y) = e^{-\pi y^2} \mapsto \Gamma(s) = e^{-\pi s^2}$

$$F(s) = \int e^{-2\pi isx} e^{-3x^2} dx = \sqrt{\frac{\pi}{3}} e^{-\pi^2 s^2/3}$$

3. Notice $f(x)$ is the inverse Fourier transform of e^{-s^4}. Hence

$$F(s) = e^{-s^4}$$

4. Complete the square and use the Fourier transform of $g(x) = \frac{1}{1+x^2}$ is $G(s) = \pi e^{-\pi|s|}$

$$F(s) = \int e^{-2\pi isx} \frac{2}{(x-2)^2 + 1} dx = 2\pi e^{-4\pi is - \pi|s|}$$

5. Use the following sequence

$$f_1(x) = \frac{1}{1+x^2} \quad \mapsto \quad F_1(s) = \pi e^{-\pi|s|} \tag{1}$$

$$f_2(x) = \frac{x}{1+x^2} = x f_1(x) \quad \mapsto \quad F_2(s) = -\frac{1}{2\pi i} \frac{d}{ds} F_1(s) = -\frac{i\pi}{2} sign(s) e^{-\pi|s|} \tag{2}$$

$$f(x) = \frac{2(x-2)}{(x-2)^2 + 1} = 2 f_2(x-2) \quad \mapsto \quad F(s) = -i\pi sign(s) e^{-4\pi is - \pi|s|} \tag{3}$$

where $sign(s) = 1$ for $s > 0$ and $sign(s) = -1$ for $s < 0$. At $s = 0$,

$$F(0) = \int_{-\infty}^{\infty} f(x) dx = 2 \int_{-\infty}^{\infty} \frac{x-2}{(x-2)^2 + 1} dx = 2 \lim_{R \to \infty} \int_{-R}^{R} \frac{y}{y^2 + 1} dy = 0$$

(the last integral is zero because $y/(y^2+1)$ is an odd function). Thus we get $F(s) = -i\pi sign(s) e^{-4\pi is - \pi|s|}$ where $sign(0) = 0$.

Note. As an alternative to computing $F(0)$ explicitely, one may invoque the Dirichlet theorem of pointwise convergence and regularize F at 0 by $F(0) = 0.5(F(0-) + F(0+))$.

II.

6. We look for a function $f : R \to C$ that satisfies the given equation. Since we have the freedom to impose any condition we want, we make a choice $f(-x) = f(x)$ (that is, we look for an even function). Then its Fourier transform is:

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} e^{-2\pi isx} f(x) dx = \int_0^{\infty} (e^{-2\pi isx} f(x) + e^{2\pi isx} f(-x)) dx \\ &= \int_0^{\infty} (e^{-2\pi isx} + e^{2\pi isx} f(x)) dx = 2 \int_0^{\infty} cos(2\pi xs) f(x) dx \end{aligned}$$

Note also:

$$F(-s) = F(s)$$

We want to have $F(s) = 2e^{-s}$ for $s > 0$. Then, because of the symmetry $F(-s) = F(s)$, we obtain $F(s) = 2e^{-|s|}$. The function f is now obtained through the synthesis equation (the inverse Fourier transform):

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi isx} F(s)ds = 2\int_{-\infty}^{\infty} e^{2\pi isx} e^{-|s|}ds = \frac{4}{1+4\pi^2x^2}$$

Note that indeed $f(-x) = f(x)$ as we assumed.

7. We are asked to find a function $F : R \to C$ that satisfies the given equation. We choose f to be odd (anti-symmetric), that is $f(-x) = -f(x)$. Then:

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} e^{-2\pi isx} f(x)dx = \\ & \int_0^{\infty} (e^{-2\pi isx} f(x) + e^{2\pi isx} f(-x))dx = -2i \int_0^{\infty} f(x)sin(2\pi sx)dx \end{aligned}$$

Note, by direct computation, that:

$$F(-s) = 2i \int_0^{\infty} f(x)sin(2\pi sx)dx = -F(s)$$

Thus the Fourier transform F must satisfy:

$$F(s) = \begin{cases} -2i & , \quad 0 < x < 1 \\ 2i & , \quad -1 < x < 0 \\ 0 & , \quad |x| > 1 \end{cases}$$

(and the regularization at -1,0,1). The function f is obtained by computing the inverse Fourier transform:

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi isx} F(s)ds = 2\frac{1 - cos(\pi x)}{\pi x}$$

Note $f(-x) = -f(x)$ as assumed.

8. Solution 1: Use Parseval identity to transform the left hand side into:

$$\int f(t)f(x-t)dt = \int F(s)F(s)e^{2\pi isx}ds = \mathcal{F}((F(s))^2)(x)$$

Hence the left hand side is the Fourier transform of $(F(s))^2$ at x. Apply inverse Fourier transform to obtain:

$$(F(s))^2 = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi isx}dx = e^{-\pi s^2}$$

Thus $F(s) = e^{-\pi s^2/2}$ from where you obtain $F(s) = \sqrt{2}e^{-2\pi x^2}$.

Solution 2: The integral is a convolution product of f with itself. Thus we seek an f so that:

$$f \star f = g \quad , \quad g(x) = e^{-\pi x^2}$$

Applying Fourier transform on both sides, we obtain:

$$F(s)^2 = e^{-\pi s^2}$$

Hence

$$F(s) = e^{-\pi s^2/2}$$

and the function f is obtain by inverse Fourier transform:

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi isx} F(s)ds = \sqrt{2}e^{-2\pi x^2}.$$

III.

9. Applying Fourier transform on both sides we get:

$$-(2\pi i s)^2 F(s) + F(s) = G(s)$$

Hence

$$F(s) = \frac{G(s)}{1+4\pi^2 s^2} = G(s)\frac{1}{1+4\pi^2 s^2}$$

Recalling the Fourier transform of a convolution is a product of the two Fourier transforms, we obtain:

$$f(x) = (g \star h)(x)$$

where h is the function whose Fourier transform is $\frac{1}{1+4\pi^2 s^2}$. Recall from previous homework that the Fourier transform of $h(x) = e^{-|x|}$ is exactly $H(s) = 2/(1+4\pi^2 s^2)$. Hence

$$f(x) = \frac{1}{2}(g \star e^{-|\cdot|})(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-u|} g(u)du$$

10. When $g = \Pi(x)$ we obtain:

$$f(x) = \frac{1}{2}\int_{-1/2}^{1/2} e^{-|x-u|} du$$

Explicitely, f is given by

$$f(x) = \begin{cases} \frac{1}{2}\int_{-1/2}^{1/2} e^{-(u-x)} du = Ce^{x} & if \quad x \leq -0.5 \\ \frac{1}{2}\int_{-1/2}^{x} e^{-(x-u)} du + \frac{1}{2}\int_{x}^{1/2} e^{-(u-x)} du = 1 - \frac{1}{2}e^{-0.5}(e^{x} + e^{-x}) & if \quad -1/2 < x < 1/2 \\ \frac{1}{2}\int_{-1/2}^{1/2} e^{-(x-u)} du = Ce^{-x} & if \quad x \geq 0.5 \end{cases}$$

where $C = \frac{1}{2}(e^{0.5} - e^{-0.5})$. Note: $f(x) \geq 0$, $f(0) = \int_0^{0.5} e^{-u} du = 1 - e^{-0.5}$, $f(-x) = f(x)$ (because $F(-s) = F(s)$), and f achieves its maximum at $x = 0$. $f'(0) = 0$. For large x, $f(x) < e^{-(x-0.5)}$, hence f decays exponentially fast. A sketch is inserted below.