1. Over $[0,1]$ the Fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}.$$

Note

$$f(x) = sin^2(3\pi x) = \frac{1}{2}(1 - cos(6\pi x)) = \frac{1}{2} - \frac{1}{4}e^{6\pi i x} - \frac{1}{4}e^{-6\pi i x}$$

Thus the Fourier coefficients are:

$$c_n = \begin{cases} \frac{1}{2} & for \quad n = 0 \\ -\frac{1}{4} & for \quad n = \pm 3 \\ 0 & otherwise \end{cases}$$

The Fourier series converges everywhere to $f(x)$.

2. Note first

$$I = \int_0^\infty x^2 cos(4\pi x) e^{-x^2} dx = \frac{1}{2}\int_0^\infty e^{4\pi i x} x^2 e^{-x^2} dx + \frac{1}{2}\int_0^\infty e^{-4\pi i x} x^2 e^{-x^2} dx = \frac{1}{2}\int_{-\infty}^\infty e^{-4\pi i x} x^2 e^{-x^2} dx$$

From

$$e^{-\pi s^2} = \int_{-\infty}^\infty e^{-2\pi i x s} e^{-\pi x^2} dx$$

substitute $y = \sqrt{\pi}x$ and $u = s/\sqrt{\pi}$ to obtain

$$\sqrt{\pi} e^{-\pi^2 u^2} = \int_{-\infty}^\infty e^{-2\pi i y u} e^{-y^2} dy$$

Differentiate two times with respect to u and divide by 2π each time:

$$1^{st}\ time\ :\ \sqrt{\pi}(-\pi u) e^{-\pi^2 u^2} = (-i)\int_{-\infty}^\infty e^{-2\pi i y u} y e^{-y^2} dy$$

$$2^{nd}\ time\ :\ \sqrt{\pi}(u^2\pi^2 - \frac{1}{2}) e^{-\pi^2 u^2} = (-i)^2\int_{-\infty}^\infty e^{-2\pi i y u} y^2 e^{-y^2} dy$$

The integral I is obtained by setting $u = 2$ in the above identity and then dividing by 2:

$$I = -\frac{1}{2}\sqrt{\pi}(4\pi^2 - \frac{1}{2}) e^{-4\pi^2}$$

3. Use the Fourier transform with respect to x:

$$u(t,x) = \int_{-\infty}^\infty e^{2\pi i x s} U(t,s) ds$$

We obtain

$$\frac{\partial U}{\partial t} = 2(2\pi i s) U$$

Thus

$$U(t,s) = e^{4\pi i s t} U(0,s)$$

and

$$u(t,x) = \int_{-\infty}^\infty e^{2\pi i (x+2t) s} U(0,s) ds = u(0, x+2t)$$

Or:

$$u(t,x) = sin(3x + 6t)(x+2t)^{27} e^{-(x+2t)^2}$$

4. We have

$$\hat{f}(s) = \sum_{n=-100}^{100} \delta(s-n) - \sum_{k=-100}^{100} e^{2\pi i k s} = -f(s)$$

5. The derivatives are

$$f'(x) = \begin{cases} 2x-3 & , \quad x<1 \\ 3-2x & , \quad 1<x<2 \\ 2x-3 & , \quad 2<x \end{cases}$$

$$f''(x) = 2\delta(x-1) + 2\delta(x-2) + \begin{cases} 2 & , \quad x<1 \\ -2 & , \quad 1<x<2 \\ 2 & , \quad 2<x \end{cases}$$

$$f'''(x) = 2\delta'(x-1) + 2\delta'(x-2) - 4\delta(x-1) + 4\delta(x-2)$$

6. Apply Fourier transform and obtain:

$$\frac{2}{1+4\pi^2 s^2} F(s) = 1$$

(note: $(x^2+x+1)\delta(x) = \delta(x)$). Hence

$$F(s) = \frac{1}{2} + 2\pi^2 s^2$$

and

$$f(x) = \frac{1}{2}\delta(x) - \frac{1}{2}\delta''(x)$$

7.
Theory I. An example of such a function:

$$f(x) = \begin{cases} 1 & for \quad 0 \leq x \leq 1 \\ 0 & for \quad 1 < x < 2 \end{cases}$$

Theory II. Maximum overshoot is 9% of jump which for $f(x)$ is $\frac{4}{3} = f(1/3+) - f(1/3-)$. Hence we get 0.12

Signal Processing I: DCT. Since the DCT transform of a constant vector is a vector with the first component nonzero followed by zero entries, it follows the DCT of a uniform gray image is a matrix of zeros with a single nonzero entry (upper left corner). Thus the 3-DCT coefficient compressed image is identical to the original one and the reconstruction error is zero.

Signal Processing II: STFT. The algorithm scales linearly with the data size. Hence for 200,000 samples it takes about 6,000,000 flops (6Mflops).

Signal Processing III: DFT vs. FFT. The matrix vector multiplication takes $O(N^2)$ flops, hence the computational complexity scales with N^2: time for $N = 10,000$ is $100 * 10ms = 1s$.

Signal Processing IV: Reverberation. Received signal is

$$u_R(t) = 0.5u(t-0.01) + 0.1u(t-0.03) = (0.5e^{-20\pi i} + 0.1e^{-60\pi i})e^{2000\pi i t} = 0.6e^{2000\pi i t}$$

Hence the amplitude is 0.6.

PDE I,II,III: Wave equation. The initial condition of the third problem is the sum between the initial condition of the first problem and of the second problem. Hence

$$u(t,x) = u_1(t,x) - \frac{1}{2}u_2(t,x) = e^{-(x-2t)^2} + e^{-(x+2t)^2} + ln(1+(x-2t)^2) - ln(1+(x+2t)^2)$$

PDE IV, V: Periodic heat equation. Note the initial condition of the second problem is exactly the solution of the first problem evaluated at $t = 1$. Thus

$$u(t,x) = u_1(t+1,x) = e^{-4(t+1)} sin(2\pi x) + e^{-36(t+1)} cos(6\pi x)$$

PDE Vi,VII: Heat equation. Since $u(t,x) = 0$ is a solution of the PDE and satisfies the initial condition, and by uniqueness of the solution, it follows $u(t,x) = 0$ is the solution of the problem.