Notes for Math 464

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Chapter 1

Review and Preliminary

In this chapter we review some necessary background from advanced calculus, and linear algebra. We also introduce a few notions that will be used throughout these notes.

1.1 Complex numbers

Definition 1.1.1. A complex number z is a number of the form $z = x + iy$ where, $x, y \in \mathbb{R}$ and i is the number such that

$$i^2 = -1.$$

The set of all complex numbers is denoted by $\mathbb{C}$.

Conjugate of a complex number: If $z = x+iy$ is a complex number, its *conjugate* is the complex number denoted $\bar{z}$ and given by $\bar{z} = x - iy$.
Operations on complex numbers: Let $z_k = x_k + iy_k$ be complex numbers for $k = 1, 2$. Then,

(i) $z_1 + z_2 = z_2 + z_1 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$,

(ii) $z_1 z_2 = z_2 z_1 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$.

(iii) In particular, if $z = x + iy$ then $z\bar{z} = x^2 + y^2 \geq 0$.

We can use (iii) above to define the *modulus (or absolute value)* of z to be the nonnegative number given by

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

Polar form and geometric interpretation of a complex number: Every complex number has *a polar form* given by

$$z = x + iy = re^{i\theta}$$

where $r = |z| = \sqrt{x^2 + y^2}$ and θ is determined by the equations $x = r\cos\theta, y = r\sin\theta$. In particular,

$$z = x + iy = r\cos\theta + ir\sin\theta.$$

It follows that to every complex number $z = x + iy$ one can associate a point P in the $xy-$ plane with coordinates $P = (x, y)$. In addition, the polar form of z is equivalent to the fact that $OP = r = |z| = \sqrt{x^2 + y^2}$ and OP makes an angle θ with the positive $x-$axis.

Example 1.1.1. 1. $z = 2 - 3i$. What is $\bar{z}, |z|$ and what is the polar form of z? Sketch z in the complex plane.

2. Unit length complex numbers and the unit circle.

1.2 Inner product space

1.2.1 Finite dimensional inner product spaces

Given $N \geq 1$ an integer, we shall denote $\mathbb{K}^N$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the vector space of all N tuples. That is $x \in \mathbb{K}^N$ if and only if $x = (x_1, x_2, \ldots, x_N)$ where each $x_k \in \mathbb{K}$.

We define on $\mathbb{K}^N$ the following operation

$$\langle x, y\rangle = \sum_{k=1}^{N} x_k \bar{y}_k$$

where $x, y \in \mathbb{K}$. This is called a *scalar product* or an *inner product* on $\mathbb{K}^N$. The inner product gives rise to a norm on $\mathbb{K}^N$ namely: for each $x = (x_1, x_2, \ldots, x_N) \in \mathbb{K}^N$,

$$\|x\| = \sqrt{\sum_{k=1}^{N} |x_k|^2}.$$

When equipped with this inner product, $\mathbb{K}^N$ is called an *inner product space.* The inner product have the following properties: let $x, y, z \in \mathbb{K}^N$ and $a, b \in \mathbb{K}$.

(i) $\langle x, y\rangle = \overline{\langle y, x\rangle}$. (Remark on $\mathbb{K} = \mathbb{R}$).

(ii) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$,

(iii) $\langle x, ay + bz\rangle = \overline{a}\langle x, y\rangle + \overline{b}\langle x, z\rangle$,

(iv) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$, We use the norm derived from the inner product to define a distance on $\mathbb{K}^N$: given $x, y \in \mathbb{K}^N$, the distance between x, and y is

$$\|x - y\| = \sqrt{\sum_{k=1}^{N} |x_k - y_k|^2}.$$

Theorem 1.2.1. *For any $x, y \in \mathbb{K}^N$, we have*
Schwartz inequality

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

with equality if and only if x or y is a nonnegative multiple of the other.
Triangle inequality

$$\|x + y\| \leq \|x\| + \|y\|$$

with equality if and only if x or y is a nonnegative multiple of the other.

Proof. Give the details for $\mathbb{K} = \mathbb{R}$.

□

Orthogonality and orthonormal basis $x, y \in \mathbb{K}^N$ are said to be *orthogonal* if and only if

$$\langle x, y\rangle = 0.$$

Given a subspace $E \subset \mathbb{K}^N$, its *orthogonal complement* is the subspace of $\mathbb{K}^N$ denoted by $E^{\perp}$ and given by

$$E^{\perp} = \{x \in \mathbb{K}^N : \langle x, y\rangle = 0 \,\text{for all}\, y \in E\}.$$

Exercise 1.2.1. Prove that $E^{\perp}$ is a subspace of $\mathbb{K}^N$ is E is one.

A set of vectors $\{e_k\}_{k=1}^N$ is an orthonormal basis for $\mathbb{K}^N$ is and only if $\{e_k\}_{k=1}^N$ is a basis and

$$\langle e_k, e_l\rangle = \delta(k - l)$$

where δ is the Kronecker delta sequence equal 1 for $k = 0$ and 0 else.

A set of vectors $\{e_k\}_{k=1}^N \subset \mathbb{K}^N$ is a basis for $\mathbb{K}^N$ if and only if it is linearly independent set and spans $\mathbb{K}^N$. This is equivalent to saying that every $x \in \mathbb{K}^N$ has a unique decomposition

$$x = \sum_{k=1}^{N} c_k e_k$$

where the coefficients $c_k \in \mathbb{K}$ are unique. Moreover, when $\{e_k\}_{k=1}^N$ is an orthonormal basis, then the coefficients are given by

$$c_k = \langle x, e_k\rangle.$$

Note that if $\{e_k\}_{k=1}^N$ is an ONB for $\mathbb{K}^N$, and if $x \in \mathbb{K}^N$ then

$$\|x\|_2^2 = \sum_{k=1}^N |\langle x, e_k\rangle|^2.$$

Moreover, if $x = \sum_{k=1}^N \langle x, e_k\rangle e_k$ and $y = \sum_{k=1}^N \langle y, e_k\rangle e_k$ then

$$\langle x, y\rangle = \sum_{k=1}^N \langle x, e_k\rangle \overline{\langle y, e_k\rangle}.$$

Orthogonal projections Given $1 \leq M \leq N$

$$P_M x = \sum_{k=1}^M \langle x, e_k\rangle e_k$$

is the orthogonal projection of $E_M = span\{e_k, k = 1, 2, \dots, M\}$.

Exercise 1.2.2. In dimension $N = 5$ give examples of two distinct ONB and write down the matrices corresponding to some orthogonal projections.

Given any basis $\{u_k\}_{k=1}^N$ for $\mathbb{K}^N$, there exists an algorithm *the Gram-Schmidt* orthogonalization procedure that transform this basis to an ONB $\{e_k\}_{k=1}^N$. In particular,

$$e_1 = u_1/\|u_1\|, e_2 = \tfrac{u_2 - \langle u_2, e_1\rangle e_1}{\|u_2 - \langle u_2, e_1\rangle e_1\|},$$

and having constructed e_l, then,

$$e_{l+1} = \tfrac{u_{l+1} - \sum_{k=1}^l \langle u_{l+1}, e_k\rangle e_k}{\|u_{l+1} - \sum_{k=1}^l \langle u_{l+1}, e_k\rangle e_k\|}.$$

1.2.2 The space $L^2([a, b])$

For $a, b \in \mathbb{R}$, consider the functions defined on (a, b) (we allow $a = -\infty$ and/or $b = \infty$). We assume throughout that all functions are continuous on (a, b) except may be at finitely many points.

The space $L^2([a, b])$ is defined by

$$L^2([a, b]) = \{f : [a, b] \to \mathbb{C} : \int_a^b |f(x)|^2 \, dx < \infty\}.$$

The integral in the definition is a Riemann integral, and when one of the bounds or both are infinite, the integral is to be interpreted as an indefinite Riemann integral. We can equipped the space with an inner product that will make it into an inner product space. In particular, for $f, g \in L^2([a, b])$, then

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)} \, dx,$$

and

$$\|f\|_{L^2([a,b])} = \|f\|_2 := \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b |f(x)|^2\,dx}.$$

Note that if f is piecewise continuous and $\|f\|_2 = 0$, then f might not be identically equal to 0. So we make the assumption that $f = g$ if f, and g are piecewise continuous and equal except at finitely many points of $[a, b]$.

Example 1.2.1. $f(x) = 1$ for $x = 0, 1$ and $f(x) = 0$ else. Then $\|f\|_2 = 0$ though f is not identically 0.

The inner product on $L^2([a, b])$ can be seen as a "natural extension of the inner product on $\mathbb{C}^N$ using Riemann's sums. For instance on $L^2([0, 1])$ we have. Given $f, g \in L^2([0, 1])$ which we assume continuous, consider for each positive integer N the vectors

$$f_N = (f(1/N), f(2/N), \dots, f(1))\; g_N = (g(1/N), g(2/N), \dots, g(1)).$$

Then

$$\lim_{N\to\infty} \tfrac{1}{N}\langle f_N, g_N\rangle = \lim_{N\to\infty} \tfrac{1}{N}\sum_{k=1}^{N} f(k/N)\overline{g(k/N)} = \int_0^1 f(x)\overline{g(x)}\,dx = \langle f, g\rangle.$$

We now consider the special case in which $a = 0$, and $b = 1$. $L^2([0, 1])$ is an infinite dimensional space. To see this, notice that the functions $1, x, x^2, \dots, x^n, \dots,$ for all $n \geq 1$ belong to $L^2([0, 1])$ and are linearly independent **Exercise: Prove this.** It is true that in $L^2([0, 1])$ there exists a sequence of function $\{f_k\}_{k=1}^{\infty}$ such that $\langle f_k, f_l\rangle = \delta(k - l)$ that is the sequence is an orthonormal system and for each $f \in L^2([0, 1])$,

$$f = \sum_{k=1}^{\infty}\langle f, f_k\rangle f_k$$

where the series convergence will be clarified soon. In this case, the sequence $\{f_k\}_{k=1}^{\infty}$ is an orthonormal basis for $L^2([0, 1])$. In the next chapter will shall construct examples of ONB for $L^2([0, 1])$.

Given an ONB $\{f_k\}_{k=1}^{\infty}$, and $M \geq 1$, then

$$P_M f(x) = f_M(x) = \sum_{k=1}^{M}\langle f, f_k\rangle f_k(x)$$

is the orthogonal projection onto the span of the first M basis vectors. In particular,

$$f = \sum_{k=1}^{\infty}\langle f, f_k\rangle f_k$$

means that $\lim_{M\to\infty} f_M = f$, that is for each $\epsilon > 0$, there exists $M_0 > 0$ such that for all $M \geq M_0$,

$$\|f_M - f\|_2 < \epsilon.$$

If $\{f_k\}_{k=1}^{\infty}$ is an ONB for $L^2([0,1])$, and if $f, g \in L^2([0,1])$ then

$$\|f\|_2^2 = \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2$$

and

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx = \sum_{k=1}^{\infty}\langle f, f_k\rangle\overline{\langle g, f_k\rangle}.$$

The function $\chi_{[0,1]}$ which equals 1 on $[0,1]$ and 0 everywhere else is an element of $L^2([0,1])$ and

$$\|\chi_{[0,1]}\| = 1.$$

Given $f \in L^2([0,1])$, then

$$\int_0^1 |f(x)|dx = \int_0^1 \chi_{[0,1]}(x)|f(x)|dx \leq \sqrt{\int_0^1 |\chi_{[0,1]}(x)|^2\,dx}\sqrt{\int_0^1 |f(x)|^2\,dx} = \|\chi_{[0,1]}\|_2\|f\|_2 = \|f\|_2.$$

This proves that if we let

$$L^1([0,1]) = \{f : [0,1] \to \mathbb{C} : \int_0^1 |f(x)|dx < \infty\}$$

and equipped this space with the norm

$$\|f\|_{L^1([0,1])} = \|f\|_1 = \int_0^1 |f(x)|dx$$

then we have

$$\|f\|_1 \leq \|f\|_2$$

meaning that

$$L^2([0,1]) \subset L^1([0,1]).$$

The proof of this fact was given above and uses an important inequality on $L^2([0,1])$ called the Cauchy-Schwartz inequality:

Theorem 1.2.2. *For all $f, g \in L^2([a,b])$ we have*

$$|\langle f, g\rangle| = |\int_a^b f(x)\,\overline{g(x)}\,dx| \leq \|f\|_2\|g\|_2.$$

Proof. Highlight of the proof. □

Functions in $L^1((a,b))$ are called *integrable functions.*

Exercise 1.2.3. Let $f(x) = \ln x, g(x) = \frac{1}{1+x^2}, h(x) = \frac{1}{\sqrt{x}}, k(x) = \frac{1}{x}$ for $x \in (0,1)$. Prove that f, g, h belong to $L^1(0,1)$.

Prove that $k \notin L^1(0,1)$.

For which values of p is the function $f_p(x) = \frac{1}{x^p}$ $x \in (0,1)$ in $L^1(0,1)$?

For which values of p is the function $f_p(x) = \frac{1}{x^p}$ $x \in (0,1)$ in $L^2(0,1)$?

Convergence in L^2 versus uniform convergence Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined from (a,b) into $\mathbb{C}$. The sequence is said to converge to a function f defined on (a,b) if the sequence of numbers $\{f_n(x)\}_{n=1}^{\infty}$ converges to $f(x)$ for each $x \in (a,b)$. More specifically, for each $x \in (a,b)$ and each $\epsilon > 0$, there is an integer $N_0 = N_0(x,\epsilon) \geq 1$ such that for all $n \geq N_0$,

$$|f_n(x) - f(x)| < \epsilon.$$

If the indice N_0 can be chosen independently of $x \in (a,b)$ then we said that f_n converges to f uniformly on (a,b). In particular, this means that for each $\epsilon > 0$, there is an integer $N_0 = N_0(\epsilon) \geq 1$ such that for all $n \geq N_0$, and for each $x \in (a,b)$ we have

$$|f_n(x) - f(x)| < \epsilon.$$

If each of the function f_n and f belongs to $L^2(a,b)$ the the sequence converges to f in L^2 if and only if for each $\epsilon > 0$ there is an integer $N_0 \geq 1$ such that for all $n \geq N_0$ we have

$$\|f_n - f\|_2 = \sqrt{\int_a^b |f_n(x) - f(x)|^2\,dx} < \epsilon.$$

Remark 1.2.1. Remark on the relation between the three types of convergence defined above.

1.2.3 The space $\ell^2(\mathbb{Z})$

Another infinite dimensional inner product that space that we shall encounter later is a space of infinite sequences given by

$$\ell^2(\mathbb{Z}) = \{a = (a_n)_{n=-\infty}^{\infty} : a_n \in \mathbb{C} \forall n \in \mathbb{Z}, \sum_{n=-\infty}^{\infty} |a_n|^2 < \infty\}.$$

An inner product on $\ell^2(\mathbb{Z})$ is defined by: for $a = (a_n)_{n=-\infty}^{\infty}, b = (b_n)_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$ set

$$\langle a, b \rangle = \sum_{n=-\infty}^{\infty} a_n \overline{b_n}.$$

This leads to the following norm:

$$\|a\|_{\ell^2(\mathbb{Z})} = \|a\|_2 = \sqrt{\sum_{n=-\infty}^{\infty} |a_n|^2}.$$

Note that to check if a sequence $a = (a_n)_{n=-\infty}^{\infty}$ belongs to $\ell^2(\mathbb{Z})$ we must check if the series

$$\sum_{n=-\infty}^{\infty} |a_n|^2$$

converges. This is a series whose general term is nonnegative. We can make appeal to the convergence theorem for nonnegative series!

Chapter 2

Fourier Series

2.1 Motivation

To motivate this material consider the following problem: **add the physical interpretation of the equation justifies the following.**

Exercise 2.1.1. Let $u(t, x)$ represent the temperature at time $t \geq 0$ and position $x \in [0, 1]$ on a piece of wire of length 1 unit. Thus, $u(t, x)$ is a function of two variable: time $t \in [0, \infty)$ and space $x \in [0, 1]$. Assume that $u(t, x)$ satisfies the following equation:

$$\begin{cases} u_t(t,x) &= u_{xx}(t,x) \qquad t > 0, 0 \leq x \leq 1 \\ u(0,x) &= f(x) \qquad 0 \leq x \leq 1 \\ u(t,0) &= 0 \\ u(t,1) &= 0. \end{cases}$$

Where f is a function defined on $[0, 1]$, u_t is the first partial derivative of u with respect to t and $u_x x$ is the second partial derivative of u with respect to x. Find an expression for $u(t, x)$ in terms of f, x and t.

Solution 2.1.1. First we assume that the solution $u(t, x)$ can be written as $u(t, x) = T(t)X(x)$ where T is only function of time t and X is only function of space x. By substituting this for of $u(t, x)$ in the original equation we obtain:

$$T'(t)X(x) = T(t)X''(x)$$

which is equivalent to

$$\tfrac{T'(t)}{T(t)} = \tfrac{X''(x)}{X(x)} \qquad \forall t > 0, \qquad x \in [0, 1].$$

This is only possible if there is a constant c such that

$$\tfrac{T'(t)}{T(t)} = \tfrac{X''(x)}{X(x)} = c \qquad \forall t > 0, \qquad x \in [0, 1].$$

Now we can solve $\frac{T'(t)}{T(t)} = c$ and get $T(t) = Ce^{ct}$ for all $t > 0$. The constant c must be negative, otherwise the temperature $u(t,x)$ will grow without bound. Thus, $T(t) = Ce^{ct}$ for $t \geq 0$, and $c < 0$.

The second equation now becomes $X''(x) = cX(x)$ where $x \in [0,1]$. This leads to $X(x) = a\cos\sqrt{-c}x + b\sin\sqrt{-c}x$ for some constants a, b. However, the initial conditions now read $T(0)X(x) = f(x)$ and $T(t)X(0) = 0 = T(t)X(1)$ for all $x \in [0,1]$ and $t > 0$. Hence, $X(0) = X(1) = 0$ which implies that $a = 0$, and $b\sin\sqrt{-c} = 0$.

If $b = 0$ we will only have the trivial solution, thus $\sin\sqrt{-c} = 0$ which implies that $\sqrt{-c} = k\pi$, where $k \in \mathbb{N}$. That is, $X_k(x) = b\sin k\pi x$ and so $u(t,x) = Cbe^{-k^2\pi^2 t}\sin k\pi x$.

By the superposition principle, any solution to the above equation is given by

$$u(t,x) = \sum_{k=1}^{\infty} b_k e^{-k^2\pi^2 t}\sin k\pi x.$$

Using the last initial condition we see that

$$u(0,x) = \sum_{k=1}^{\infty} c_k \sin k\pi x = f(x).$$

So the equation will have a solution if the function f can be expressed as an infinite series :

$$f(x) = \sum_{k=1}^{\infty} c_k \sin k\pi x.$$

This is an example of a Fourier series.

Some questions come to mind: How are the coefficients c_k computed and how are they related to f? What type of convergence does the series possess? These are some of the questions we shall address below.

2.2 Fourier series on $[0,1]$

2.2.1 Periodic functions

Definition 2.2.1. Let $T > 0$ be a positive real number. A function $f : \mathbb{R} \to \mathbb{R}$ is $T-$*periodic* if

$$f(x+T) = f(x)$$

for all $x \in \mathbb{R}$. In this case, f is completely determined by its values on any interval of length T such as $[a, a+T)$ for any $a \in \mathbb{R}$. In the sequel we shall consider any $T-$periodic function to be defined on the interval $[0,T)$. The real number T is called a period of f. Notice that $2T, 3T, \ldots, nT$ are all period of f. The smallest period will be called *the period* of f.

Example 2.2.1. $f(x) = \sin x$, $g(x) = \cos x$ are both $2\pi-$periodic. $h(x) = \sin 2x$ is $\pi-$periodic. Notice that h can also be considered also as a $2\pi-$periodic function.

2.2.2 Fourier series for $1-$periodic function

Definition 2.2.2. Let $f : [0, 1) \to \mathbb{R}$ be a $1-$periodic function. The following series is called the Fourier series associated to f:

$$f(x) \approx a_0 + \sum_{k=1}^{\infty} a_k \cos 2\pi kx + b_k \sin 2\pi kx \tag{2.1}$$

for some coefficients a_0, a_k, b_k for $k = 1, 2, \ldots$.

Lemma 2.2.1. *For each $k, \ell \in \mathbb{N}$ we have*

$$\begin{cases} \int_0^1 \sin 2k\pi x \cos 2\ell\pi x \, dx &= 0 \qquad \forall k, \ell \geq 1 \\ \int_0^1 \sin 2k\pi x \sin 2\ell\pi x \, dx &= \frac{1}{2}\delta(k-\ell) \\ \int_0^1 \cos 2k\pi x \cos 2\ell\pi x \, dx &= \frac{1}{2}\delta(k-\ell) \\ \int_0^1 \cos 2k\pi x \, dx &= 0 \\ \int_0^1 \sin 2k\pi x \, dx &= 0, \end{cases} \tag{2.2}$$

where δ is the sequence defined by $\delta(k) = 0$ for all $k \neq 0$ and $\delta(0) = 1$.

Proof. Direct integration. □

The above lemma can be summarized as saying that the family of functions

$$\{1, \sqrt{2}\cos 2k\pi x, \sqrt{2}\sin 2k\pi x\}_{k=1}^{\infty} = \{1, \sqrt{2}\cos 2\pi x, \sqrt{2}\sin 2\pi x, \sqrt{2}\cos 4\pi x, \sqrt{2}\sin 4\pi x, \ldots\}$$

is an orthonormal set in $L^2[0, 1)$.

Using Lemma 2.2.1 we can prove:

Theorem 2.2.1. *If $f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos 2k\pi x + b_k \sin 2k\pi x$, then*

$$a_0 = \int_0^1 f(x)\, dx$$

$$a_k = 2\int_0^1 f(x) \cos 2k\pi x \, dx$$

$$b_k = 2\int_0^1 f(x) \sin 2k\pi x \, dx$$

Proof. Straight computations. □

Definition 2.2.3. Given $f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos 2k\pi x + b_k \sin 2k\pi x$, the coefficients a_0, a_k, b_k, $k \geq 1$ given in Theorem 2.2.1 are called the *Fourier coefficients* of f.

Example 2.2.2. Find the Fourier coefficients of $f(x) = \chi_{[0,1)}(x)$, $g(x) = x$ and $h(x) = \sin 2\pi x + 5\cos 6\pi x - 4\sin 10\pi x$ where $x \in [0, 1)$ and the functions are considered $1-$periodic.

Remark 2.2.1. From the above we can view the Fourier series of f as a *transformation* that sends the function f to its Fourier coefficients $\{a_0, a_k, b_k : k = 1, 2, \ldots\}$ which are defined in Theorem 2.2.1. More specifically, we have

$$f \leftrightarrow \{a_0, a_k, b_k : k = 1, 2, \ldots\}.$$

The question that comes to mind is the following: For which functions can one compute the Fourier coefficients and thus form a Fourier series? One can prove that for any function that is integrable on $L^1[0,1)$, the Fourier coefficients can be computed. Hence the Fourier series can be formed. The next question is whether this Fourier series converges, and if it does what is the limit? We will just touch upon certain aspect of these questions in the lecture.

2.2.3 Fourier series on other interval

Theorem 2.2.2. *If* $f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(2k\pi x/T) + b_k \sin(2k\pi x/T)$ *is* T *periodic, then*

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\,dx$$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos(2k\pi x/T)\,dx$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin(2k\pi x/T)\,dx$$

Proof. Just use a change of variable. □

Example 2.2.3. Consider the $2-$periodic function f defined by

$$f(x) = \begin{cases} x & \text{if} \quad x \in [0,1] \\ 1 & \text{if} \quad x \in [1,2). \end{cases}$$

Find the Fourier series of f.

2.2.4 Sine and Cosine Fourier series

Definition 2.2.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. f is an *even* function if $f(-x) = f(x)$ for all $x \in \mathbb{R}$. f is an *odd* function if $f(-x) = -f(x)$.

The following result about the integration of even and odd functions is easy to prove.

Lemma 2.2.2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function.*

- *If f is an even function and if $a \in \mathbb{R}$ the*

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

- *If f is an odd function and if $a \in \mathbb{R}$ the*

$$\int_{-a}^{a} f(x)dx = 0.$$

Example 2.2.4. (i) Assume that f is a $1-$periodic and even function. Find the Fourier coefficients of f.

(ii) Assume that g is a $1-$periodic and odd function. Find the Fourier coefficients of g.

Solution 2.2.1. (i) $a_0 = \int_0^1 f(x)dx = \int_{-1/2}^{1/2} f(x)dx = 2\int_0^{1/2} f(x)dx.$

$a_k = 2\int_0^1 f(x)\cos(2\pi kx)dx.$

$b_k = 2\int_0^1 f(x)\sin(2\pi kx)\,dx = 2\int_{-1/2}^{1/2} f(x)\sin(2\pi kx)\,dx = 0$ because $f(x)\sin(2\pi kx)$ is an odd function.

(ii) $a_0 = \int_0^1 g(x)dx = \int_{-1/2}^{1/2} g(x)dx = 0$ because g is odd.

$a_k = 2\int_0^1 g(x)\cos(2\pi kx)dx = 2\int_{-1/2}^{1/2} g(x)\cos(2\pi kx)\,dx = 0,$ since $g(x)\cos(2\pi kx)$ is odd.

$b_k = 2\int_0^1 g(x)\sin(2\pi kx)\,dx.$

Theorem 2.2.3. *Let f be a $1-$periodic function defined from $\mathbb{R}$ into $\mathbb{R}$.*

(i) If f is even and $\{a_0, a_k, b_k : k = 1, 2, \ldots, \infty\}$ are the Fourier coefficients of f then $b_k = 0$ for all $k \geq 1$. Consequently, the Fourier series of f reduces to

$$f(x) \approx a_0 + \sum_{k=1}^{\infty} a_k \cos 2k\pi x.$$

(ii) If f is odd and $\{a_0, a_k, b_k : k = 1, 2, \ldots, \infty\}$ are the Fourier coefficients of f then $a_k = 0$ for all $k \geq 0$. Consequently, the Fourier series of f reduces to

$$f(x) \approx \sum_{k=1}^{\infty} b_k \sin 2k\pi x.$$

Example 2.2.5. (i) Let $f : [0, 1/2) \to \mathbb{R}$ defined by $f(x) = x$. Find the even $1-$periodic extension of f, and find the Fourier series of this extension.

(ii) Let $f : [0, 1/2) \to \mathbb{R}$ defined by $f(x) = x$. Find the odd $1-$periodic extension of f, and find the Fourier series of this extension.

2.2.5 Complex form of the Fourier series

Exercise 2.2.1. For each integer $m \in \mathbb{Z}$, that is $m = 0, \pm 1, \pm 2, \pm 3, \ldots$ let $e_m(x) = e^{2\pi imx}$ for $x \in \mathbb{R}$. Thus $e_m : \mathbb{R} \to \mathbb{C}$ by $e_m(x) = e^{2\pi imx}$.
a) Prove that e_m is a $1-$periodic function.
b) Compute the inner product between e_m and e_n for any $m, n \in \mathbb{Z}$. That is what is $\langle e_m, e_n \rangle$?
c) Let $f : \mathbb{R} \to \mathbb{R}$ be a $1-$periodic function whose Fourier coefficients are a_0, a_k, b_k, $k \geq 1$. Let $c_k = \langle f, e_k \rangle$ for each $k \in \mathbb{Z}$. Find a relation between c_0 and a_0 as well as between c_k, a_k, and b_k for each $k \geq 1$.

Solution 2.2.2. a) For each $x \in \mathbb{R}$,

$$e_m(x+1) = e^{2\pi im(x+1)} = e^{2\pi imx + 2\pi im} = e^{2\pi imx} e^{2\pi im} = e^{2\pi imx} = e_m(x).$$

Hence e_m is $1-$periodic.
b)

$$\langle e_m, e_n \rangle = \int_0^1 e_m(x)\overline{e_n(x)}\, dx = \int_0^1 e^{2\pi imx} e^{-2\pi inx}\, dx = \int_0^1 e^{2\pi i(m-n)x} dx.$$

If $m \neq n$ then, $m - n \neq 0$ and

$$\int_0^1 e^{2\pi i(m-n)x} dx = \tfrac{1}{2\pi i(m-n)} e^{2\pi i(m-n)x}|_0^1 = 0.$$

If $m = n$, then

$$\int_0^1 e^{2\pi i(m-n)x} dx == \int_0^1 dx = 1.$$

c) Assume that $k > 0$

$$\begin{aligned} c_k &= \langle f, e_k \rangle \\ &= \int_0^1 f(x)\overline{e_k(x)} dx \\ &= \int_0^1 f(x) e^{-2\pi kix} dx \\ &= \int_0^1 f(x)(\cos 2\pi kx - i \sin 2\pi kx) dx \\ &= \int_0^1 f(x) \cos 2\pi kx dx - i \int_0^1 f(x) \sin 2\pi kx dx \\ &= \tfrac{a_k}{2} - i \tfrac{b_k}{2}. \end{aligned}$$

If $k < 0$, the $-k > 0$ and

$$c_k = \langle f, e_k \rangle = \int_0^1 f(x) e^{-2\pi ikx} dx = \int_0^1 f(x) \overline{e^{-2\pi i(-k)x}} dx$$

hence,

$$c_k = \overline{\int_0^1 f(x)e^{-2\pi i(-k)x}dx} = \overline{\langle f, e_{-k}\rangle} = \tfrac{a_{-k}}{2} + i\tfrac{b_{-k}}{2}.$$

$$c_0 = \langle f, e_0\rangle = \int_0^1 f(x)dx = a_0.$$

Note that for $k \geq 1$, $\overline{c_k} = c_{-k}$ and so:

$$a_k = c_k + \overline{c_k} = c_k + c_{-k} \,\text{and}\, b_k = i(c_k - \overline{c_k}) = i(c_k - c_{-k}).$$

Theorem 2.2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a $1-$periodic function. If $f(x) \approx \sum_{k=-\infty}^{\infty} c_k e^{2\pi kx}$ for $x \in [0, 1)$ then*

$$c_k = \int_0^1 f(x)e^{-2\pi ikx}dx,$$

and $\{c_k : k = 0, \pm1, \pm2, \pm3, \ldots\}$ are called the complex Fourier coefficients *of f. Moreover, these complex Fourier coefficients of f are related to the Fourier coefficients a_0, a_k, b_k by:*

$$c_0 = a_0, c_k = \tfrac{a_k}{2} - i\tfrac{b_k}{2}$$

when $k \geq 1$. For $k \leq -1$, $c_k = \overline{c_{-k}}$.

2.3 Convergence of Fourier series

2.3.1 Partial sums of a Fourier series

Throughout this section we assume that $f : \mathbb{R} \to \mathbb{R}$ is $1-$periodic. Denote by

$$F(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos 2\pi kx + b_k \sin 2\pi kx$$

where a_0, a_k, b_k are the Fourier coefficients defined in Section 2. For each $K \geq 1$ we define the Kth partial sum of this Fourier series by

$$F_K(x) = a_0 + \sum_{k=1}^{K} a_k \cos 2\pi kx + b_k \sin 2\pi kx \tag{2.3}$$

Our goal is to give condition on the function f such that $F(x) := \lim_{K\to\infty} F_K(x)$ exists and equals to $F(x) = f(x)$ for $x \in [0, 1)$.

Remark 2.3.1. Note that for any $1-$periodic function f, if $\int_0^1 |f(x)|dx < \infty$, then the Fourier coefficients a_0, a_k, b_k or c_k can be computed, and thus the Fourier series $F(x)$ given above can be formed. Thus the Fourier series of every function in $L^1([0, 1))$ is well defined.

Question: What about a function in $L^2[0, 1)$?

2.3.2 Riemann-Lebesgue lemma

Theorem 2.3.1. *Suppose f is a $T-$periodic and piecewise continuous function, then*

$$\lim_{k\to\infty}\int_0^T f(x)\cos(2\pi kx/T)\,dx = \lim_{k\to\infty}\int_0^T f(x)\sin(2\pi kx/T)\,dx = \lim_{k\to\infty}\int_0^T f(x)e^{-2\pi ikx/T}\,dx = 0.$$

Proof. First assume that $f(x) = \sum_{n=1}^N d_n\chi_{[a_n,b_n)}(x)$ is $1-$periodic, and compute its Fourier coefficients. Notice for each $n = 1,2,\dots,N$ we have $c_k(\chi_{[a_n,b_n)}) = \int_{a_n}^{b_n} e^{-2\pi kx}dx = \frac{e^{2\pi ka_n}-e^{-2\pi ikb_n}}{2\pi ik}$. Clearly, this sequence converges to 0 as $k\to\infty$ and so will any finite combination of such functions.

The remaining part of the proof consists of approximating any piecewise smooth function uniformly by simple functions as above. □

Theorem 2.3.2. *Let f be a $1-$periodic function. Assume that f' exists and is continuous except at finitely many points. Let c_k denote the (complex) Fourier coefficients of f, and denote by c_k' the Fourier coefficients of f'. Then for each $k = \pm1,\pm2,\pm3,\dots$*

$$c_k' = (2\pi ik)c_k.$$

More generally, if $f^{(n)}$ exists and is continuous except at finitely many points in $[0,1)$, and if $c_k^{(n)}$ denotes the Fourier coefficients of $f^{(n)}(x)$ then

$$c_k^{(n)} = (2\pi ik)^n c_k$$

for each $k = \pm1,\pm2,\pm3,\dots$.

Proof. Use integration by parts. □

2.3.3 Pointwise convergence of Fourier series

Theorem 2.3.3. *Suppose that f is continuous and $1-$periodic. At each point x where f is differentiable, the Fourier series of f converges to $f(x)$, that is*

$$\lim_{K\to\infty} a_0 + \sum_{k=1}^K a_k\cos 2\pi kx + \sin 2\pi kx = f(x).$$

To prove the theorem we need some preparations.

Lemma 2.3.1. *For each $x\in[0,1)$ let*

$$P_N(u) = 1 + 2\cos 2\pi u + 2\cos 4\pi u + 2\cos 6\pi u + \ldots + 2\cos 2N\pi u.$$

Then,

$$P_N(u) = \begin{cases} \frac{\sin(2N+1)\pi u}{\sin\pi u} & : \quad u\neq 0 \\ 2N+1 & : \quad u = 0 \end{cases}$$

Moreover,

$$\int_0^1 P_N(u)du = 1.$$

Proof. Based on geometric sums. □

Proof. : Proof of Theorem 2.3.3 Consider the $N-th$ partial Fourier sum:

$$F_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos 2\pi nx + b_n \sin 2\pi nx.$$

$$\begin{aligned}
F_N(x) &= a_0 + \sum_{n=1}^{N} a_n \cos 2\pi nx + b_n \sin 2\pi nx \\
&= \int_0^1 f(u)du + \sum_{n=1}^{N} 2\int_0^1 f(u)\cos 2\pi nudu\ \cos 2\pi nx + 2\int_0^1 f(u)\sin 2\pi nudu\ \sin 2\pi nx \\
&= \int_0^1 (1 + 2\sum_{n=1}^{N} \cos 2\pi nu \cos 2\pi nx + \sin 2\pi nu \sin 2\pi nx) f(u)\, du \\
&= \int_0^1 (1 + 2\sum_{n=1}^{N} \cos 2\pi n(u-x)) f(u)du \\
&= \int_0^1 P_N(x-u) f(u)du \\
&= \int_x^{x-1} P_N(u) f(u+x)du \\
&= \int_0^1 P_N(u) f(x+u)du.
\end{aligned}$$

Now look at

$$\begin{aligned}
f(x) - F_N(x) &= \int_0^1 f(x)P_N(u)du - \int_0^1 P_N(u)f(x+u)du \\
&= \int_0^1 (f(x) - f(x+u))P_N(u)du \\
&\int_0^1 \tfrac{f(x)-f(x+u)}{\sin \pi u} \sin(2N+1)\pi udu.
\end{aligned}$$

Now if we denote bu $g_x(u)$ the function

$$g_x(u) = \tfrac{f(x)-f(x+u)}{\sin \pi u},$$

it is clear that g_x is well defined an continuous for $u \neq 0$. Now

$$\lim_{u\to 0} g_x(u) = \lim_{u\to 0} \tfrac{f(x)-f(x+u)}{\sin \pi u} = -\pi f'(x)$$

. Thus g_x is defined and continuous for all u and by Riemann-Lebesgue lemma we have

$$\lim_{N\to\infty} f(x) - F_N(x) = \lim_{N\to\infty} \int_0^1 \tfrac{f(x)-f(x+u)}{\sin \pi u} \sin(2N+1)\pi u du = 0.$$

$\square$

In fact, a following stronger result holds. Recall that a piecewise smooth function is a continuous function that is differentiable everywhere except possibly for a discrete set of points.

Theorem 2.3.4. *If f is $1-$periodic and piecewise smooth on $[0,1)$ then the Fourier series of F converges uniformly to f on $[0,1)$.*

Remark 2.3.2. What happens at a point where f is not continuous? If f has a (finite) jump discontinuity and if f is right-differentiable and left-differentiable, then we can still use the above proof with some modifications.

Example 2.3.1. a) Let $f(x)$ be the even, $1-$periodic function given by $f(x) = x$ for $x \in [0, 1/2)$. Prove that the Fourier series of f converges to f at each point $x \in [-1/2, 1/2)$. Does the Fourier series of f converge uniformly?
b) Use part a to find the values of $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

2.4 Fourier series in $[0,1)$: the L^2 theory

Recall that

$$\{1, \sqrt{2}\cos 2\pi x, \sqrt{2}\sin 2\pi x, \ldots\} = \{1, \sqrt{2}\cos 2\pi kx, \sqrt{2}\sin 2\pi kx : k = 1, 2, \ldots\}$$

is an orthonormal set in $L^2[0,1)$, and so is

$$\{e^{2\pi inx} : n = 0, \pm 1, \pm 2, \pm 3, \ldots\}.$$

Note also that $L^2[0,1) \subset L^1[0,1)$ and so it makes sense to compute the Fourier coefficients of any L^2 function.

Theorem 2.4.1. *Let $f \in L^2[0,1)$, and*

$$f_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos 2\pi nx + b_n \sin 2\pi nx$$

where a_0, a_k, b_k are the Fourier coefficients of f. Then f_N converges to f in $L^2[0,1)$ that is $\|f_N - f\|_2 \to 0$ as $N \to \infty$.

Similarly, if

$$f_N(x) \sum_{n=-N}^{N} c_n e^{2\pi inx}$$

where c_n are the complex Fourier coefficients of f, then f_N converges to f in $L^2[0,1)$.

The following are two very important results about the L^2 theory of Fourier series.

Theorem 2.4.2. *Let*

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos 2\pi kx + b_k \sin 2\pi kx \in L^2[0,1)$$

Then

$$\int_0^1 |f(x)|^2\, dx = |a_0|^2 + \tfrac{1}{2} \sum_{k=1}^{\infty} |a_k|^2 + |b_k|^2.$$

Similarly, if

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi nix} \in L^2[0,1)$$

then

$$\int_0^1 |f(x)|^2\, dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

2.5 Other properties of the Fourier coefficients

2.5.1 Convolution

Definition 2.5.1. Let f, g be two $1-$periodic function. The *convolution* of f and g is the function defined by

$$f * g(x) = g * f(x) = \int_0^1 f(t)g(x-t)dt.$$

Theorem 2.5.1. *Let f, g be two $1-$periodic and integrable functions. The $f * g$ is $1-$periodic and its Fourier coefficients are given by*

$$c_n(f * g) = c_n(f)c_n(g),$$

for each $n = 0, \pm 1, \pm 2, \ldots$.

Proof.

$$\begin{aligned}
c_n(f*g) &= \int_0^1 f*g(x)e^{-2\pi inx}dx \\
&= \int_0^1 \int_0^1 f(t)g(x-t)dt\, e^{-2\pi inx}\, dx \\
&= \int_0^1 f(t) \int_0^1 g(x-t)e^{-2\pi inx}dxdt \\
&= \int_0^1 f(t) \int_{-t}^{1-t} g(y)e^{-2\pi in(y+t)}dydt \\
&= \int_0^1 f(t)e^{-2\pi int} \int_0^1 g(y)e^{-2\pi iny}dydt \\
&= \int_0^1 f(t)e^{-2\pi int}c_n(g)dt \\
&= c_n(f)c_n(g)
\end{aligned}$$

□

2.5.2 Other properties

Exercise 2.5.1. Let $f : [0,1) \to \mathbb{C}$ be a $1-$periodic function with Fourier coefficients c_n. Find the Fourier coefficients of $\overline{f}$, $f(-x)$, and $\overline{f(-x)}$. What can conclusions can be drawn?

Solution 2.5.1.

$$c_n(\overline{f}) = \int_0^1 \overline{f(x)}e^{-2\pi inx}dx = \overline{\int_0^1 f(x)e^{2\pi inx}dx} = \overline{c_{-n}(f)}.$$

$$\begin{aligned}
c_n(f(-x)) &= \int_0^1 f(-x)e^{-2\pi inx}dx \\
&= -\int_0^{-1} f(x)e^{2\pi inx}dx \\
&= \int_{-1}^0 f(x)e^{2\pi inx}dx \\
&= \int_0^1 f(x)e^{2\pi inx}dx \\
&= c_{-n}(f) \\
&= \overline{c_n(\overline{f})}
\end{aligned}$$

Note that if f is real-valued, then $\overline{f(x)} = f(x)$ and so in the last formula we have

$$c_n(f(-x)) = c_{-n}(f) = \overline{c_n(f)}.$$

Combining the last two properties we see that

$$c_n(\overline{f(-x)}) = \overline{c_n(f)} = \overline{c_{-n}(f)} = \overline{c_n(\overline{f})}.$$

From this exercise we get the following results:

Proposition 2.5.1. *let $f : [0,1) \to \mathbb{C}$ be $1-$periodic.*

(i) If f is real-valued then

$$c_n(f(-x)) = c_{-n}(f) = \overline{c_n(f)}.$$

In addition, if $f(-x) = f(x)$ then,

$$c_n(f) = c_n(f(-x)) = c_{-n}(f) = \overline{c_n(f)}.$$

(ii) If f is complexed valued and $f(-x) = f(x)$ then

$$c_n(f(-x)) = c_n(f) == c_{-n}(f) = \overline{c_n(\overline{f})}.$$

Chapter 3

Fourier transform

3.1 Definition and examples

3.1.1 Definition

Definition 3.1.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\int_{-\infty}^{\infty} |f(x)|dx < \infty$. The *Fourier transform* of f is the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\gamma}dx. \tag{3.1}$$

The *inverse Fourier transform* is given by

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\gamma)e^{2\pi x i\gamma}d\gamma. \tag{3.2}$$

Remark 3.1.1. (i.) Notice that for any real-valued function f, the Fourier transform is complex-valued.

(ii.) The Fourier transform of f is defined as long as the indefinite integral

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty.$$

The space of all such function will be denoted

$$L^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : \int_{-\infty}^{\infty} |f(x)|dx < \infty\}.$$

This is a vector space on which one can define a norm by: for $f \in L^1(\mathbb{R})$,

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(x)|dx.$$

Question: Give example of functions in $L^1(\mathbb{R})$.

Proposition 3.1.1. *If $f \in L^1(\mathbb{R})$, then its Fourier transform $\hat{f}$ is a bounded and uniformly continuous function on $\mathbb{R}$. In particular,*

$$|\hat{f}(\gamma)| \leq \|f\|_1$$

for all $\gamma \in \mathbb{R}$.

Proof. The fact that $\hat{f}$ is bounded follows from

$$|\hat{f}(\gamma)| = |\int_{-\infty}^{\infty} f(x)e^{-2\pi i x\gamma}dx| \leq \int_{-\infty}^{\infty} |f(x)e^{-2\pi i x\gamma}|dx = \int_{-\infty}^{\infty} |f(x)|dx = \|f\|_1$$

for each $\gamma \in \mathbb{R}$.

To prove that $\hat{f}$ is uniformly continuous, first assume that f is smooth and equal 0 outside an interval $[-a, a]$. Then, for $\gamma, \gamma' \in \mathbb{R}$

$$\begin{aligned}
|\hat{f}(\gamma) - \hat{f}(\gamma')| &= |\int_{-\infty}^{\infty} f(x)(e^{-2\pi i\gamma x} - e^{-2\pi i\gamma' x})dx| \\
&= |\int_{-a}^{a} f(x)(e^{-2\pi i\gamma x} - e^{-2\pi i\gamma' x})dx| \\
&\leq \int_{-a}^{a} |f(x)(e^{-2\pi i\gamma x} - e^{-2\pi i\gamma' x})|dx \\
&\leq \int_{-a}^{a} |f(x)||(e^{-2\pi i\gamma x} - e^{-2\pi i\gamma' x})|dx \\
&= \int_{-a}^{a} |f(x)||(e^{-2\pi i\gamma x} - e^{-2\pi i\gamma' x})|dx \\
&= \sqrt{2}\int_{-a}^{a} |f(x)||\sin(\pi x(\gamma - \gamma'))|dx \\
&\leq \sqrt{2}\pi|\gamma - \gamma'|\int_{-a}^{a} |xf(x)|dx \\
&\leq \sqrt{2}a\pi|\gamma - \gamma'|\|f\|_1.
\end{aligned}$$

The proof for general $f \in L^1(\mathbb{R})$ uses the fact that any such function f can be approximated with a smooth function that is 0 outside an interval $[-a, a]$ for some $a > 0$. □

Let

$$L^\infty(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} : \sup_{x\in\mathbb{R}} |f(x)| < \infty\}$$

then L^∞ is a vector space equipped with the norm

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|.$$

It follows that the Fourier transform is a map

$$\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$$

given by

$$\mathcal{F}(f)(\gamma) = \hat{f}(\gamma).$$

The inverse Fourier transform will be seen as the inverse $\mathcal{F}^{-1}$ of this map. We will describe the properties of the map $\mathcal{F}$ in the next section.

Exercise 3.1.1. Justify informally why $\mathcal{F}^{-1}$ is the inverse of $\mathcal{F}$.

3.1.2 Examples

Let f be defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 1 & : \quad |x| \leq 1/2 \\ 0 & : \quad |x| > 1/2 \end{cases}$$

For $\gamma \in \mathbb{R}$, and $\gamma \neq 0$,

$$\begin{aligned} \hat{f}(\gamma) &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\gamma} \\ &= \int_{-1/2}^{1/2} 1 e^{-2\pi x\gamma} dx \\ &= \tfrac{-1}{2\pi i\gamma} e^{-2\pi i x\gamma}\big|_{x=-1/2}^{x=1/2} \\ &= \tfrac{-1}{2\pi i\gamma}(e^{-\pi i\gamma} - e^{\pi i\gamma}) \\ &= \tfrac{e^{\pi i\gamma} - e^{-\pi i\gamma}}{2\pi i\gamma} \\ &= \tfrac{2i\sin\pi\gamma}{2\pi i\gamma} \\ &= \tfrac{\sin\pi\gamma}{\pi\gamma} \end{aligned}$$

For $\gamma = 0$ we have

$$\hat{f}(0) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x 0} dx = \int_{-1/2}^{1/2} 1 dx = 1$$

Therefore,

$$\hat{f}(\gamma) = \begin{cases} \frac{\sin\pi\gamma}{\pi\gamma} & : \quad \gamma \neq 0 \\ 1 & : \quad \gamma = 0 \end{cases}$$

Notice that

$$\lim_{\gamma\to 0} \hat{f}(\gamma) = \lim_{\gamma\to 0} \tfrac{\sin\pi\gamma}{\pi\gamma} = 1 = \hat{f}(0).$$

So $\hat{f}$ is a continuous function on $\mathbb{R}$. The function $\hat{f}(\gamma) = \frac{\sin \pi\gamma}{\pi\gamma}$ is called the *sinc* function and is denoted:

$$sinc(\gamma) = \tfrac{\sin \pi\gamma}{\pi\gamma}.$$

For the second example let: $g(x) = f * f(x)$, where f is defined above. Then

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} f(t)f(x-t)dt \\ &= \int_{-1/2}^{1/2} f(x-t)dt \\ &= -\int_{1/2+x}^{x-1/2} f(y)dy \\ &= \int_{x-1/2}^{x+1/2} f(y)dy \end{aligned}$$

If $x + 1/2 \leq -1/2$, that is, $x \leq -1$ or if $x - 1/2 \geq 1/2$, that is $x \geq 1$, then $g(x) = 0$, as the integrand in the last integral vanishes. So we must consider only x such that $|x| \leq 1$. If $-1 < x < 0$, then $-1/2 < x + 1/2 < 1/2$ and $-3/2 < x - 1/2 < -1/2$, so the last integral reduces to

$$g(x) = \int_{-1/2}^{x+1/2} f(y)dy = \int_{-1/2}^{x+1/2} dy = x + 1/2 + 1/2 = x + 1.$$

If $0 < x < 1/2$, $1/2 < x + 1/2 < 1$ and $-1/2 < x - 1/2 < 0$, so the integral defining g becomes

$$g(x) = \int_{x-1/2}^{1/2} f(y)dy = \int_{x-1/2}^{1/2} dy = 1/2 - x + 1/2 = 1 - x.$$

Therefore,

$$g(x) = \begin{cases} 1 - |x| & : \ |x| \leq 1 \\ 0 & : \ |x| > 1 \end{cases}$$

The Fourier transform of g is now given by: Let $\gamma \neq 0$.

$$\begin{aligned}
\hat{g}(\gamma) &= \int_{-\infty}^{\infty} g(x)e^{-2\pi i\gamma x}dx \\
&= \int_{-1}^{1} (1-|x|)e^{-2\pi i\gamma x}dx \\
&= \int_{-1}^{0} (1+x)e^{-2\pi i\gamma x}dx + \int_{0}^{1} (1-x)e^{-2\pi i\gamma x}dx \\
&= \tfrac{-(1+x)e^{-2\pi\gamma x}}{2\pi i\gamma} + \tfrac{e^{-2\pi i\gamma x}}{4\pi^2\gamma^2}|_{-1}^{0} + \tfrac{-(1-x)e^{-2\pi\gamma x}}{2\pi i\gamma} - \tfrac{e^{-2\pi i\gamma x}}{4\pi^2\gamma^2}|_{0}^{1} \\
&= \tfrac{2-2\cos 2\pi\gamma}{4\pi^2\gamma^2} \\
&= \tfrac{4\sin^2 \pi\gamma}{4\pi^2\gamma^2} \\
&= \tfrac{\sin^2 \pi\gamma}{\pi^2\gamma^2}
\end{aligned}$$

$$\hat{g}(0) = \int_{-1}^{1} (1-|x|)dx = 1.$$

We will compute the Fourier transform of $h(x) = e^{-\pi x^2}$, in the next subsection.

3.2 Properties of the Fourier transform

Theorem 3.2.1. *For any $f \in L^1(\mathbb{R})$,*

$$\lim_{|\gamma|\to\infty} \hat{f}(\gamma) = 0.$$

Proof. This is the Riemann-Lebesgue lemma. To prove it, first assume that f is continuous and $f(x) = 0$ for all $|x| > A$ for some large $A > 0$. This f can in turn be approximated by a piecewise constant function, for which the theorem holds. □

From now on we will view the Fourier transform as an *operator* that is a function whose domain is a subspace of functions. This operator was denoted by $\mathcal{F}$. We now list some properties of this operator.

Theorem 3.2.2. *Let f and g be defined on $\mathbb{R}$ such that f is smooth and $f(x) = 0$ for $|x|$ large. Assume that $g \in L^1(\mathbb{R})$. Then the following hold:*

(i) $\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g)$ and for any constant a, $\mathcal{F}(af) = a\mathcal{F}(f)$. We say in this case that $\mathcal{F}$ is a linear operator.

(ii) $\mathcal{F}(x^n f(x))(\gamma) = \frac{1}{(-2\pi i)^n}\frac{d^n}{d\gamma^n}\mathcal{F}(f)(\gamma)$

(iii) For any constant a, $\mathcal{F}(f(x-a))(\gamma) = e^{-2\pi i a\gamma}\mathcal{F}(f)(\gamma)$.

(iv) For any constant a, $\mathcal{F}(e^{2\pi iax}f(x))(\gamma) = \mathcal{F}(f)(\gamma - a)$.

(v) For any $b \in \mathbb{R}$, $\mathcal{F}(f(bx))(\gamma) = \frac{1}{b}\mathcal{F}(f)(\gamma/b)$.

(vi) For any n, $\mathcal{F}(f^{(n)})(\gamma) = (2\pi i\gamma)^n \mathcal{F}(f)(\gamma)$.

Exercise 3.2.1. Find the Fourier transform of $h(x) = e^{-\pi x^2}$, by proving that $\hat{h}$ is the solution to $y' + 2\pi\gamma y = 0$ and $y(0) = 1$.

Solution 3.2.1.

$$\begin{aligned}
\hat{h}'(\gamma) &= \int_{-\infty}^{\infty} e^{-\pi x^2}(-2\pi ix)e^{-2\pi ix\gamma}dx \\
&= i\int_{-\infty}^{\infty}(-2\pi xe^{-\pi x^2})e^{-2\pi ix\gamma}dx \\
&= ie^{-\pi x^2}e^{-2\pi ix\gamma}|_{-\infty}^{\infty} - i\int_{-\infty}^{\infty} e^{-\pi x^2}(-2\pi i\gamma)e^{-2\pi ix\gamma}dx \\
&= 2\pi i^2\gamma\int_{-\infty}^{\infty} e^{-\pi x^2}e^{-2\pi ix\gamma}dx \\
&= -2\pi\gamma\hat{h}(\gamma)
\end{aligned}$$

which shows that

$$\hat{h}'(\gamma) + 2\pi\gamma\hat{(h)}(\gamma) = 0.$$

But $\hat{h}(0) = \int_{\infty}^{\infty} e^{-\pi x^2}dx = 1$. Thus, $\hat{h}(\gamma)$ is the unique solution to the given initial value problem. But this is a linear first order equation and its solution is $y = e^{-\pi\gamma^2}$. Therefore, $\hat{h}(\gamma) = e^{-\pi\gamma^2}$.

Theorem 3.2.3. *If f, g are integrable functions, then $f * g$ defined by*

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} g(y)f(x-y)dy = g * f(x)$$

is integrable and

$$f \hat{*} g(\gamma) = \hat{f}(\gamma)\hat{g}(\gamma).$$

Proof.

$$\begin{aligned}
f \hat{*} g(\gamma) &= \int_{-\infty}^{\infty} f * g(x) e^{-2\pi i x\gamma} dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(x-y) dy e^{-2\pi i x\gamma} dx \\
&= \iint_{-\infty}^{\infty} f(y) g(x-y) e^{-2\pi i x\gamma} dy dx \\
&= \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} g(x-y) e^{-2\pi i x\gamma} dx dy \\
&= \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} g(u) e^{-2\pi i (u+y)\gamma} du dy \\
&= \int_{-\infty}^{\infty} f(y) e^{-2\pi i \gamma y} \int_{-\infty}^{\infty} g(u) e^{-2\pi i u\gamma} du dy \\
&= \int_{-\infty}^{\infty} f(y) e^{-2\pi i \gamma y} \hat{f}(\gamma) dy \\
&= \hat{f}(\gamma) \int_{-\infty}^{\infty} f(y) e^{-2\pi i \gamma y} dy \\
&= \hat{f}(\gamma)\hat{g}(\gamma).
\end{aligned}$$

□

3.3 L^2 theory of the Fourier transform

Definition 3.3.1. Let $f \in L^2(\mathbb{R})$, be given. The Fourier transform of f is the $L^2(\mathbb{R})$ function $\hat{f}$ that is defined by

$$\hat{f}(\gamma) = \lim_{n\to\infty} \int_{-n}^{n} f(x) e^{-2\pi i x\gamma} dx$$

where the limit is taken in the L^2 sense. That is

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |\hat{(f)}(\gamma) - \int_{-n}^{n} f(x) e^{-2\pi i x\gamma} dx|^2 d\gamma = 0.$$

Theorem 3.3.1. *If $f, g \in L^2(\mathbb{R})$, then the following hold:*

(i) $\langle \hat{f}, g \rangle = \langle \mathcal{F}(f), g \rangle_{L^2} = \langle f, \mathcal{F}^{-1}(g) \rangle_{L^2}$.

(ii) $\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$.

In particular,

$$\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$$

Note that the fact that $\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$ is called Plancherel theorem and $\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$ is Parseval identity.

Example 3.3.1. Let f be defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 1 & : \quad |x| \leq 1/2 \\ 0 & : \quad |x| > 1/2 \end{cases}$$

Prove that

$$\|\hat{f}\|_{L^2} = \|sinc\gamma\|_{L^2} = 1.$$

3.4 The Heisenberg Uncertainty principle

Definition 3.4.1. Given a function $f \in L^2(\mathbb{R})$, we define the *dispersion* of f about a point $a \in \mathbb{R}$ by

$$\Delta_a f = \frac{\int_{-\infty}^{\infty} (x-a)^2 |f(x)|^2 \, dx}{\|f\|^2_{L^2(\mathbb{R})}}.$$

Note that if

$$\|f\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |f(x)|^2 dx = 1,$$

then $|f(x)|^2 dx$ can be interpreted as a probability distribution of $\mathbb{R}$ and $\Delta_a f$ is just the variance of this probability distribution when $a = \int_{-\infty}^{\infty} x|f(x)|^2 dx$.

Theorem 3.4.1. *Given any $a, b \in \mathbb{R}$, we have*

$$\Delta_a f \Delta_b \hat{f} = \frac{\int_{-\infty}^{\infty} (x-a)^2 |f(x)|^2 \, dx}{\|f\|^2_{L^2(\mathbb{R})}} \frac{\int_{-\infty}^{\infty} (\gamma-b)^2 |\hat{f}(\gamma)|^2 \, d\gamma}{\|\hat{f}\|^2_{L^2(\mathbb{R})}} \geq \frac{1}{16\pi^2}$$

for all $f \in L^2(\mathbb{R})$.

If we take $a = \int_{-\infty}^{\infty} x|f(x)|^2 dx$, and $b = \int_{-\infty}^{\infty} \gamma|\hat{f}(\gamma)|^2 d\gamma$, and if we assume that $\|f\|_{L^2} = \|\hat{f}\|_{L^2} = 1$, then the Heisenberg uncertainty principle reduces to

$$\int_{-\infty}^{\infty} (x-a)^2 |f(x)|^2 \, dx \int_{-\infty}^{\infty} (\gamma - b)^2 |\hat{f}(\gamma)|^2 \, d\gamma \geq \frac{1}{16\pi^2}.$$

In particular, $\Delta_a f$ and $\Delta_b \hat{f}$ cannot be made small simultaneously. Therefore, if a function is well localized in space (or time) ($x-$variable), then it cannot be well-localized in frequency (γ-variable).

3.5 Linear time-invariant transform

We consider transforms (that is functions acting on other functions) that are defined on the vector space of all piecewise continuous functions.

Definition 3.5.1. A transform L is linear if $L(f+g) = L(f) + L(g)$, and $L(af) = aL(f)$ for any f, g and any scalar a.

A linear transform is time-invariant if $L[f(x-a)] = L[f](x-a)$ for every function f and any scalar a. This means that the image under L of a delayed signal, is the same as the delayed image $L(f)$.

Example 3.5.1. Prove that $L[f](x) = \int_0^x f(s)ds$ is linear but not time-invariant.

Prove that $L[f](x) = h * f(x)$ where $h \in L^1(\mathbb{R})$ is a linear time-invariant transformation.

Theorem 3.5.1. *If L is a linear time invariant transformation on the space of piecewise continuous functions, and if $\gamma \in \mathbb{R}$ is fixed, then there exists a function $h : \mathbb{R} \to \mathbb{C}$ such that*

$$L(e^{2\pi i\gamma x}) = \hat{h}(\gamma)e^{2\pi i\gamma x}.$$

In addition,

$$L(f) = h * f.$$

Proof. We only sketch the proof of this result. Let $a \in \mathbb{R}$ be fixed and let $\gamma \in \mathbb{R}$. Set $g_\gamma(x) = e^{2\pi i\gamma x}$, and $g_{\gamma,a}(x) = g_\gamma(x-a) = e^{2\pi i\gamma(x-a)}$. Using the fact that L is time-invariant we can write:

$L(g_{\gamma,a})(x) = L(g_\gamma)(x-a)$. On the other hand, $g_{\gamma,a}(x) = e^{-2\pi i\gamma}e^{2\pi i\gamma x} = e^{-2\pi i\gamma a}g_\gamma(x)$. Using the fact that L is linear we can write: $L(g_{\gamma,a})(x) = e^{-2\pi ia\gamma}L(g_\gamma)(x)$. Consequently,

$$L(g_{\gamma,a})(x) = L(g_\gamma)(x-a) = e^{-2\pi ia\gamma}L(g_\gamma)(x).$$

Since this holds for arbitrary a, if we choose $x = a$ we see that

$$L(g_\gamma)(0) = e^{-2\pi i\gamma x}L(g_\gamma)(x)$$

and so

$$L(g_\gamma)(x) = L(g_\gamma)(0)e^{2\pi i\gamma x}.$$

We can define the function h by letting $\hat{h}(\gamma) = L(g_\gamma)(0)$. This completes the proof of the first part.

For the second part, we use the fact that L is linear and $f(x) = \int_{-\infty}^{\infty} \hat{f}(\gamma)e^{2\pi i\gamma x}dx$ to write

$$L(f)(x) = \int_{-\infty}^{\infty} \hat{f}(\gamma)L(e^{2\pi i\gamma x})d\gamma = \int_{-\infty}^{\infty} \hat{f}(\gamma)\hat{h}(\gamma)e^{2\pi i\gamma x}d\gamma = h * f(x).$$

Note that we have used the fact that the integral defining the inverse Fourier transform can be approximated using (finte) Riemann sums. □

The function $h(x)$ defined in Theorem 3.5.1 is called *the impulse response function* of the linear time-invariant transformation L.

Example 3.5.2. Find the impulse response of each of the following linear time-invariant transformations:

(i) L_1 filters out all frequency above $B_1/2$ and below $-B_1 < /2$.

(ii) L_3 retains only frequencies γ such that $B_1 \leq |\gamma| \leq B_2$

, where B_1, B_2 are some positive numbers.

3.6 The Sampling Theorem

Definition 3.6.1. A function $f \in L^2(\mathbb{R})$ is said to be *band-limited* is there exists $B > 0$, such that

$$\hat{f}(\gamma) = 0 \qquad |\gamma| > B/2.$$

The smallest number B, for which the last equation holds, is called the *bandwidth* of f.

Example 3.6.1. (i) $f(x) = \frac{\sin \pi x}{\pi x}$

(ii) $g(x) = e^{-\pi x^2} * \frac{\sin \pi x}{\pi x}$

Remark 3.6.1. A function $f \in L^2(\mathbb{R})$ is called *time-limited* if there exists $T > 0$, such that

$$f(x) = 0 \qquad |x| > T/2.$$

Note that by the Heisenberg uncertainty principle, it is impossible for a non-zero function to be both band-limited and time-limited.

The following theorem is known as the Shannon-Whittaker Sampling Theorem

Theorem 3.6.1. *Let f be a band-limited function with bandwidth $B > 0$. f is completely determine by its values at the points $\frac{k}{B}$, for $k = 0, \pm 1, \pm 2, \ldots$. In particular,*

$$f(x) = \sum_{k=-\infty}^{\infty} f(k/B) sincB(x - k/B) = \sum_{k=-\infty}^{\infty} f(k/B) \frac{\sin \pi B(x-k/B)}{\pi B(x-k/B)} \tag{3.3}$$

where the series converges uniformly.

Proof. Write the (complex) Fourier series of $\hat{f}(\gamma)$ considered as a $B-$periodic function. □

Lemma 3.6.1. *Given $B > 0$, we denote the space of band-limited functions with bandwidth B, by*

$$PW(B) = \{f \in L^2(\mathbb{R}) : \hat{f}(\gamma) = 0, |\gamma| > B/2\}.$$

It follows that $\{\sqrt{B} sinc[B(x - k/B)] = \sqrt{B}\frac{\sin \pi B(x-k/B)}{\pi B(x-k/B)} : k = 0, \pm 1, \pm 2, \ldots\}$ *is an orthonormal basis for $PW(B)$.*

In fact, Let f be a band-limited function with bandwidth B. If $T > 0$ is such that $0 < TB < 1$, then f can be reconstructed from its samples at kT, with $k = 0, \pm 1, \pm 2, \pm 3, \ldots$. In particular,

$$f(x) = T\sum_{k=-\infty}^{\infty} f(kT)\frac{\sin[\pi(x-kT)/T]}{\pi(x-kT)} = \sum_{k=-\infty}^{\infty} f(kT)\frac{\sin[\pi(x-kT)/T]}{[\pi(x-kT)/T]} = \sum_{k=-\infty}^{\infty} f(kT) sinc[(x-kT)/T],$$

where the series converges uniformly. The condition $TB < 1$ is equivalent to $T < 1/B$. The case $T = 1/B$ is exactly what we proved in Theorem 3.6.1. The rate of sampling $T = 1/B$ is called the *Nyquist* rate, and the condition $TB < 1$ is called the *Nyquist condition*. The conclusion we take from here is that any band-limited function can be reconstructed from its samples taken at a sampling rate satisfying the Nyquist condition.

Chapter 4

Generalized functions on $\mathbb{R}$

We would like to give a rigorous meaning to expression such as

$$\int_{-\infty}^{\infty} \phi(x)\delta(x)dx = \phi(0), \int_{-\infty}^{\infty} \phi(x)e^{-\pi x^2}dx = f\{\phi\}.$$

For what type of functions ϕ can the above expression makes sense? How do we interpret these expressions?

4.1 Schwartz functions

Definition 4.1.1. A function $\phi : \mathbb{R} \to \mathbb{C}$ is called a *Schwartz function* if ϕ is infinitely continuously differentiable, that is $\phi', \phi'', \phi^{(3)}, \ldots$ exist and are continuous and if

$$\lim_{|x|\to\infty} x^n \phi^{(m)}(x) = 0, \forall m = 0, 1, 2, \ldots, n = 0, 1, 2, \ldots.$$

The set of all Schwartz functions will be denoted $\mathbb{S}$

Example 4.1.1. If $f(x) = e^{-\pi x^2}$, then $f \in \mathbb{S}$.

Let $g(x) = \frac{\sin \pi x}{\pi x}$. Is $g \in \mathbb{S}$?

Given any $a < b$, there exists a function $\phi \in \mathbb{S}$ such that $\phi(x) > 0$ if $a < x < b$, $\phi(x) = 0$ if $x \leq a$ or $x \geq b$.

In fact, given any $a < b < c < d$, there exists a function $\phi \in \mathbb{S}$ such that $\phi(x) = 0$ for $x \leq a$ or $x \geq d$, $\phi(x) = 1$ is $b \leq x \leq c$, $\phi'(x) > 0$ for $a < x < b$ and $\phi'(x) < 0$ for $c < x < d$.

Exercise 4.1.1. Let $p \in [1, \infty]$. Prove that if $\phi \in \mathbb{S}$, then $\phi \in L^p(\mathbb{R})$, that is $\int_{-\infty}^{\infty} |\phi(x)|^p dx < \infty$ if $1 \leq p < \infty$, and $\sup_{x\in\mathbb{R}} |\phi(x)| < \infty$.

Prove that if $\phi \in \mathbb{S}$, then $\phi^{(m)} \in \mathbb{S}$ for all $m = 1, 2, \ldots$.

Prove that if $\phi \in \mathbb{S}$, then $x^n\phi(x) \in \mathbb{S}$ for all $n = 1, 2, \ldots$.

Prove that if $\phi \in \mathbb{S}$, then $x^n\phi^{(m)}(x) \in \mathbb{S}$ for all $n = 1, 2, \ldots, m = 1, 2, \ldots$.

Lemma 4.1.1. *$\mathbb{S}$ is a linear space: if $\phi_1, \phi_2 \in \S$ and $a, b \in \mathbb{C}$, then $a\phi_1 + b\phi_2 \in \mathbb{S}$.*

If $\phi_1, \phi_2 \in \mathbb{S}$, then $\phi_1 * \phi_2, \phi_1\phi_2$ all belong to $\mathbb{S}$.

Proposition 4.1.1. *If $\phi \in \mathbb{S}$, then $\hat{\phi}(\gamma) = \int_{-\infty}^{\infty} \phi(x)e^{-2\pi i\gamma x}dx \in \mathbb{S}$ and $\phi(x) = \int_{-\infty}^{\infty} \hat{\phi}(\gamma)e^{2\pi i\gamma x}d\gamma$.*

Proof. Give the proof in class. □

What is a functional? $\delta\{\phi\} = \phi(0)$ for $\phi \in \mathbb{S}$ is a "function" whose domain is $\mathbb{S}$! This is an example of functional.

Definition 4.1.2. Let f be a function defined on $\mathbb{R}$. The *fundamental functional* corresponding to f is the functional defined on $\mathbb{S}$ by:

$$f\{\phi\} = \int_{-\infty}^{\infty} f(x)\phi(x)dx.$$

Remark 4.1.1. For $\phi \in \mathbb{S}$ we define the *Dirac delta functional* by

$$\delta\{\phi\} = \phi(0) := \int_{-\infty}^{\infty} \delta(x)\phi(x)dx.$$

Definition 4.1.3. A function $f : \mathbb{R} \to \mathbb{C}$ is said to be *slowly growing* if

$$\lim_{|x|\to\infty} \tfrac{f(x)}{x^n} = 0$$

for some choice of $n = 0, 1, 2, \ldots$.

If in addition f is continuous, we will say that f is a *continuous slowly growing (CSG)* function.

Example 4.1.2. (i) Any function $f \in \mathbb{S}$ is CSG.

(ii) Any polynomial is CSG.

(iii) $\sin x, \cos x, \ln|x|, x\ln|x|$

Proposition 4.1.2. *Let g be a CSG function, and let $\phi \in \mathbb{S}$. Prove that $g\phi$ is a CSG function. Moreover, prove that $g\phi \in L^1(\mathbb{R})$.*

Given any CSG function g we will associate the fundamental functional defined by

$$g\{\phi\} := \int_{-\infty}^{\infty} g(x)\phi(x)dx,$$

for each $\phi \in \mathbb{S}$.

Assume that g is CGS and has a derivative g' which is also CSG. Then for each $\phi \in \mathbb{S}$ we have

$$\begin{aligned}\int_{-\infty}^{\infty} g'(x)\phi(x)\,dx &= \lim_{L\to-\infty,U\to\infty}\int_L^U g'(x)\phi(x)dx\\ &= \lim_{L\to-\infty,U\to\infty} g(x)\phi(x)|_L^U - \int_L^U g(x)\phi'(x)dx\\ &= -\int_{-\infty}^{\infty} g(x)\phi'(x)dx\end{aligned}$$

Consequently, we can define

$$g'\{\phi\} := -\int_{-\infty}^{\infty} g(x)\phi'(x)dx,$$

for $\phi \in \mathbb{S}$. g' will be called the *generalized derivative* of the CSG function g.

More generally, the *nth generalized derivative* of the CSG function g is defined by

$$g^{(n)}\{\phi\} := (-1)^n \int_{-\infty}^{\infty} g(x)\phi^{(n)}(x)dx,$$

for $\phi \in \mathbb{S}$.

Definition 4.1.4. We say that f is a *generalized function* if $f = g^{(n)}$ for some choice of CSG function g and for some nonnegative integer n.

Remark 4.1.2. Given two generalized functions f_1, f_2 and $a < b$ we will say that $f_1(x) = f_2(x)$ for $x \in (a, b)$ if $f_1\{\phi\} = f_2\{\phi\}$ for all $\phi \in \mathbb{S}$ with $\phi(x) = 0$ for $x < a$ or $x > b$.

If $a = -\infty$ and $b = \infty$, then $f_1(x) = f_2(x)$ for $x \in \mathbb{R}$ if $f_1\{\phi\} = f_2\{\phi\}$ for all $\phi \in \mathbb{S}$.

Example 4.1.3. Show that $f(x) = \operatorname{sgn}(x)$, $f(x) = \llcorner x \lrcorner$, and $f(x) = \ln|x|$ are generalized functions.

4.2 Common generalized functions

Find and simplify the functional that is used to represent each of the following generalized function

(a) $p_n(x) = x^n$, $n = 0, 1, 2, \ldots$,

Note that each of the functions p_n is CSG, thus for each $\phi \in \mathbb{S}$,

$$p_n\{\phi\} = \int_{-\infty}^{\infty} p_n(x)\phi(x)dx = \int_{-\infty}^{\infty} x^n\phi(x)dx.$$

Now, the generalized derivative of p_n is

$$p_n'\{\phi\} = -\int_{-\infty}^{\infty} p_n(x)\phi'(x)dx = -\int_{-\infty}^{\infty} x^n\phi'(x)dx = n\int_{-\infty}^{\infty} x^{n-1}\phi(x)dx = np_{n-1}\{\phi\}.$$

Thus, $p_n' = np_{n-1}$.

(b) $c(x) = \cos x$, $s(x) = \sin x$. For each $\phi \in \mathbb{S}$ we have

$$c\{\phi\} = \int_{-\infty}^{\infty} \cos x\phi(x)dx, \text{ and } s\{\phi\} = \int_{-\infty}^{\infty} \sin x\phi(x)dx.$$

Moreover,

$$c'\{\phi\} = -\int_{-\infty}^{\infty} \cos x\phi'(x)dx = -\int_{-\infty}^{\infty} \sin x\phi(x)dx = -s\{\phi\}$$

, and so $c' = -s$. Similarly, $s' = c$.

(c) $r, r', r'', \ldots r^{(n)}$ where

$$r(x) = \begin{cases} x & : \quad x > 0 \\ 0 & : \quad x \leq 0 \end{cases}$$

r is a CSG and so for each $\phi \in \mathbb{S}$ we have

$$r\{\phi\} = \int_{-\infty}^{\infty} r(x)\phi(x)dx = \int_{0}^{\infty} x\phi(x)dx.$$

$$r'\{\phi\} = -\int_{-\infty}^{\infty} r(x)\phi'(x)dx = -\int_{0}^{\infty} x\phi'(x)dx = \int_{0}^{\infty} \phi(x)dx := H(\phi)$$

where H is the Heaviside function defined by

$$H(x) = \begin{cases} 1 & : \quad x > 0 \\ 0 & : \quad x \leq 0 \end{cases}$$

$$r''\{\phi\} = (-1)^2\int_{-\infty}^{\infty} r(x)\phi''(x)dx = \int_{0}^{\infty} x\phi''(x)dx = -\int_{0}^{\infty} \phi'(x)dx = \phi(0) := \delta(\phi)$$

So the Dirac delta function is the generalized function given by $\delta := r''$. Note that $\delta^{(n)} = r^{(n+2)}$ and $\delta^{(n)}\{\phi\} = (-1)^n\phi^{(n)}(0)$.

(d) $q, q'q'', \ldots, q^{(n)}$ where $q(x) = \int_0^x \tau(t)dt$ with $\tau(x) = \llcorner x \lrcorner = m$ if $m \leq x < m+1$, and $m = 0, \pm 1, \pm 2, \ldots$.

τ is slowly growing but not continuous, however, q is CSG. Thus for $\phi \in \mathbb{S}$ we have $q\{\phi\} = \int_{-\infty}^{\infty} q(x)\phi(x)dx$, and

$$q'\{\phi\} = -\int_{-\infty}^{\infty} q(x)\phi'(x)dx = \sum_{m=-\infty}^{\infty} \int_{m}^{m+1} -q(x)\phi'(x)dx.$$

Now $q'(x) = m$ for $m < x < m+1$ and $m = 0, \pm 1, \pm 2, \ldots$.

$$\begin{aligned} q'\{\phi\} &= \sum_{m=-\infty}^{\infty} [q(m)\phi(m) - q(m+1)\phi(m+1) + \int_{m}^{m+1} m\phi(x)dx \\ &= \int_{\infty}^{\infty} \tau(x)\phi(x)dx := \tau\{\phi\} \end{aligned}$$

$$\begin{aligned} q''\{\phi\} &= (-1)^2 \int_{-\infty}^{\infty} q(x)\phi''(x)dx \\ &= \sum_{m=-\infty}^{\infty} \int_{m}^{m+1} -m\phi'(x)dx \\ &= \sum_{m=-\infty}^{\infty} m\phi(m) - m\phi(m+1) \\ &= \sum_{m=-\infty}^{\infty} \phi(m). \end{aligned}$$

$$comb\{\phi\} = \sum_{m=-\infty}^{\infty} \phi(m).$$

This is the Comb generalized function. It is immediate that

$$comb^{(n)}\{\phi\} = (-1)^{(n)} \sum_{m=-\infty}^{\infty} \phi^{(n)}(m).$$

(d) $\ell, \ell', \ell'', \ldots \ell^{(n)}$, where $\ell(x) = \int_0^x \ln|t|dt = x\ln|x| - x$. ℓ is CSG so if $\phi \in \mathbb{S}$ we have

$$\ell\{\phi\} = \int_{-\infty}^{\infty} \ell(x)\phi(x)dx = \int_{-\infty}^{\infty} (x\ln|x| - x)\phi(x)dx.$$

$$\ell'\{\phi\} = -\int_{-\infty}^{\infty} \ell(x)\phi'(x)dx = \int_{-\infty}^{\infty} \ln|x|\phi(x)dx.$$

Justify why $\ell'' = x^{-1} = p_{-1}$. In particular,

$$\ell''\{\phi\} = p_{-1}\{\phi\} = \lim_{L\to\infty} \int_{-L}^{L} \tfrac{\phi(x)-\phi(0)}{x} dx.$$

To prove this statement, let $\phi \in \mathbb{S}$ and define $\phi_1(x) = \phi(x) - \phi(0)$. Observe that ϕ_1 is continuous, but not in $\mathbb{S}$ (why?). Also, $\phi_1'(x) = \phi'(x)$ for all x. Now

$$\begin{aligned}
\ell''\{\phi\} &= (-1)^2 \int_{-\infty}^{\infty} (x\ln|x| - x)\phi''(x)dx \\
&= -\int_{-\infty}^{\infty} \ln|x|\phi'(x)dx \\
&= -\int_{-\infty}^{\infty} \ln|x|\phi_1'(x)dx \\
&= \lim_{L\to\infty}\lim_{\epsilon\to 0^+} -\int_{-L}^{-\epsilon} \ln|x|\phi_1'(x)dx + \int_{\epsilon}^{L} \ln|x|\phi_1'(x)dx \\
&= \lim_{L\to\infty}\lim_{\epsilon\to 0^+} [\ln|x|\phi_1(x)|_{-L}^{-\epsilon} + \int_{-L}^{-\epsilon} \tfrac{\phi_1(x)}{x}dx - \ln|x|\phi_1(x)|_{\epsilon}^{L} + \int_{\epsilon}^{L} \tfrac{\phi_1(x)}{x}dx \\
&= \lim_{L\to\infty} \int_{-L}^{L} \tfrac{\phi_1(x)}{x}dx
\end{aligned}$$

where the continuity of ϕ_1 was used and $\lim_{x\to 0} \ln|x|\phi_1(x) = 0$.

4.3 Operations on generalized functions

4.3.1 Linearity

If f_1, f_2 are two generalized function and if c_1, c_2 are two scalars, then $c_1 f_1 + c_2 f_2$ is also a generalized function. The collection of all generalized functions is a *linear space* denoted $\mathbb{G}$.

4.3.2 Translate, dilate, derivative, Fourier transform

For $x_0 \in \mathbb{R}$ and $\phi \in \mathbb{S}$ we let $T_{x_0}\phi(x) = \phi(x + x_0)$. If $a > 0$, then $S_a\phi(x) = \phi(ax)$. $D\phi(x) = \phi'(x)$, and $\mathcal{F}\phi(x) = \hat{\phi}(x)$.

4.3.3 Translate, dilate, derivation, and Fourier transform

Proposition 4.3.1. *Let f be a generalized function, let $x_0 \in \mathbb{R}, a \in \mathbb{R}$, define then $f_1(x) = T_{x_0}f(x) = f(x + x_0), f_2(x) = S_a f(x) = f(ax), f_3(x) = Df(x) = f'(x)$, and $f_4(s) = \mathcal{F}f(s) = \hat{f}(s)$ all defined generalized functions. In particular, for each $\phi \in \mathbb{S}$ we have*

$$f_1\{\phi\} = T_{x_0}f\{\phi\} = f\{T_{-x_0}\phi\} = \int_{-\infty}^{\infty} f(x)\phi(x - x_0)dx$$

$$f_2\{\phi\} = S_a f\{\phi\} = \tfrac{1}{|a|} f\{S_{1/a}\phi\} = \tfrac{1}{|a|}\int_{-\infty}^{\infty} f(x)\phi(x/a)dx$$

$$f_3\{\phi\} = Df\{\phi\} = f'\{\phi\} = -\int_{-\infty}^{\infty} f(x)\phi'(x)dx$$

$$f_4\{\phi\} = \mathcal{F}(f)\{\phi\} = \hat{f}\{\phi\} = \int_{-\infty}^{\infty} f(s)\hat{\phi}(s)ds$$

Example 4.3.1. Simplify: $\delta''(x-5), comb'(x), \hat{p_1}, \hat{\delta}, \hat{p_{-1}}$.

Remark 4.3.1. Given a function $\phi \in \mathbb{S}$ we can define the following new functions:

$$\check{\phi}(x) = \phi(-x), \phi^{\dagger}(x) = \overline{\phi(-x)}, \phi^{-}(x) = \overline{\phi(x)}.$$

We can then define similar operation of generalized functions: if $f \in \mathbb{G}$, then for each $\phi \in \mathbb{S}$ we have

$$\check{f}\{\phi\} = f\{\check{\phi}\}, f^{\dagger}\{\phi\} = f\{\phi^{\dagger}\}, f^{-}\{\phi\} = f\{\phi^{-}\}.$$

Then the generalized function f is even if $\check{f} = f$, odd if $\check{f} = -f$, real if $f^{-} = f$, pure imaginary if $f^{-} = -f$, hermitian if $f^{\dagger} = f$, and anti hermitian if $f^{\dagger} = -f$

4.3.4 Multiplication and convolution

Let α be a function on $\mathbb{R}$ such that $\alpha, \alpha', \alpha'', \ldots$ are all CSG. Then given a generalized function $f \in \mathbb{G}$, we can define the *product* of α and f to be the generalized function given by

$$[\alpha \cdot f]\{\phi\} = f\{\alpha\phi\},$$

for each $\phi \in \mathbb{S}$.

Let β be a function on $\mathbb{R}$ such that $\hat{\beta}, \hat{\beta}', \hat{\beta}'', \ldots$ are all CSG. Then given a generalized function $f \in \mathbb{G}$, we can define the *convolution* of β and f to be the generalized function given by

$$[\beta * f]\{\phi\} = f\{\phi * \check{\beta}\},$$

for each $\phi \in \mathbb{S}$.

Example 4.3.2. Find $\alpha(x) \cdot \delta(x - x_0)$

Prove that for each generalized function f, we have $\delta * f = f$.

Prove that $x \cdot p_{-1}(x) = 1$.

Prove that $\alpha \cdot \delta^{(k)}(x - x_0) = \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-\ell} \alpha^{(k-\ell)}(x_0)\delta^{(\ell)}(x - x_0)$.

If g is a CSG and $\phi \in \mathbb{S}$, $\phi * g^{(n)} = [\phi * g]^{(n)} = \phi^n * g$. In fact, if f is a generalized function given by $f = g^{(n)}$ for some CSG g and $n \geq 0$, and if $\phi \in \mathbb{S}$ we have $\phi * f = \phi^{(n)} * g, (\phi * f)' = \phi^{(n+1)} * g, \ldots$.

Exercise 4.3.1. Can the following be defined? $\delta \cdot \delta, \delta \cdot p_{-1}, 1 * 1, x * x^3$?

Is it true that $[f_1 \cdot f_2] \cdot f_3 = f_1 \cdot [f_2 \cdot f_3], [f_1 * f_2] * f_3 = f_1 * [f_2 * f_3]$? Hint consider δ, x, p_{-1} and $1, \delta'$, and $sgn(x)$.

4.3.5 Division

Let g be a generalized function and α be a function such that $\alpha, \alpha', \alpha'', \ldots$ are all CSG. If $1/\alpha, 1/\alpha', 1/\alpha'', \ldots$ are all CSG then there exists a unique generalized function f such that $\alpha \cdot f = g$. In fact, $f = \frac{1}{\alpha} \cdot g$.

Example 4.3.3. Find $f \in \mathbb{G}$ such that $e^{i\pi x^2} \cdot f(x) = e^{-\pi x^2}$.

Does there exists $f \in \mathbb{G}$ such that $e^{-\pi x^2} \cdot f(x) = 1$?

Proposition 4.3.2. *The solution to* $(x-x_0)^n \cdot f = 0$ *where* f *is a generalized function on* $\mathbb{R}$*, and* $n \geq 0$ *is*

$$f(x) = \sum_{k=0}^{n-1} c_k \delta^{(k)}(x - x_0)$$

for some coefficients c_k*.*

Example 4.3.4. Solve $(x^2 - 4) \cdot f(x) = 1$:

$\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{1}{4}[\frac{1}{x-2} - \frac{1}{x+2}]$. Therefore,

$$(x^2 - 4) \cdot \tfrac{1}{4}[\tfrac{1}{x-2} - \tfrac{1}{x+2}] = \tfrac{1}{4}[(x + 2) - (x - 2)] = 1$$

Thus the equation to be solve can be written as $(x^2 - 4) \cdot f(x) = 1 = (x^2 - 4) \cdot \frac{1}{4}[\frac{1}{x-2} - \frac{1}{x+2}]$ leading to the homogeneous equation

$$(x^2 - 4) \cdot [f(x) - \tfrac{1}{4}p_{-1}(x - 2) + \tfrac{1}{4}p_{-1}(x + 2)] = 0$$

and this can be solve easily by

$$f(x) = \tfrac{1}{4}p_{-1}(x - 2) - \tfrac{1}{4}p_{-1}(x + 2) + c_1\delta(x - 2) + c_2\delta(x + 2).$$

4.4 Derivative and simple differential equations

Rules of derivative: if $f_1, f_2, f_3,$ are generalized functions and if $\alpha, \alpha', \alpha'', \ldots,$ and $\hat{\beta}, \hat{\beta}', \hat{\beta}'', \ldots$ CSG , and if $a, x_0 \in \mathbb{R}$, $c_1, c_2 \in \mathbb{C}$ then:

$[c_1 f_1(x)+c_2 f_2(x)]' = c_1 f_1'(x)+c_2 f_2'(x)$, $[f(x-x_0)]' = f'(x-x_0)$, $[f(ax)]' = af'(ax)$, $[\alpha(x) \cdot f(x)]' = \alpha(x) \cdot f'(x) + \alpha'(x) \cdot f(x)$, $[(\beta * f)(x)]' = (\beta' * f)(x) = (\beta * f')(x)$.

Let H be the Heaviside function, prove that $r(x) = x \cdot H(x)$ where r is the ram function and use this to find r''.

$$h(x - x_1) = \begin{cases} 1 & : \quad x > x_1 \\ 0 & : \quad x < x_1 \end{cases}$$

Find f'.

Assume that f has the jump $J_k = f(x_k^+) - f(x_k^-)$ at the point x_k, $k = 1, 2, \cdots, m$ the piecewise smooth function $f_0(x) = f(x) - \sum_{k=1}^{m} J_k h(x - x_k)$ is continuous. And so $f'(x) = f_0'(x) + \sum_{k=1}^{m} J_k \delta(x - x_k)$ with f_0' being represented by the fundamental functional of the ordinary derivative.

Find $f', f'', \ldots$ if $f(x) = e^{-x}H(x)$.