# Fourier Transform: Additional Rules and Examples.

$$f : \mathbb{R} \to \mathbb{C} \xrightarrow{F.T.} F(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} f(x)\, dx$$

**Recall:** Plancherel Formula:

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

$$\boxed{\text{Reconstruction Formula / Inversion Formula / Synthesis Formula}}$$

**Question:** How to recover $f$ from $F$?

**Answer:**

**Theorem** Plancherel/Parseval

① Assume $F \in L^2(\mathbb{R})$, $\int_{-\infty}^{\infty} |F(s)|^2 ds < \infty$, or equivalently, $f \in L^2(\mathbb{R})$

Then:

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i x s} \cdot F(s)\, ds.$$

(in $L^2$-sense).

with convergence in $L^2$-sense (or, mean-square convergence):

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \left| f(x) - \int_{-R}^{R} e^{2\pi i x s} F(s)\, ds \right|^2 dx = 0.$$

② [Dirichlet]. Assume $f$ satisfies "some regularity conditions". Then for every $x \in \mathbb{R}$

$$\lim_{R\to\infty} \int_{-R}^{R} e^{2\pi i x s} F(s)\, ds = \frac{1}{2}\left(f(x-0) + f(x+0)\right).$$

(pointwise result/reconstruction)

Why: Plancherel $\Rightarrow$ Inversion.

1. $$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx \overset{(*)}{\Longrightarrow}$$

$\Rightarrow$ If $f, g \in L^2(\mathbb{R})$ and $F, G$ are their Fourier transform:

$$\boxed{\int_{-\infty}^{\infty} F(s)\cdot \overline{G(s)}\, ds = \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx.}$$ (Parseval Formula)

(polarization identity).

(idea: use $(*)$ for
$\|f+g\|^2 = \|F+G\|^2$
$\|f-g\|^2 = \|F-G\|^2$
$\|f+ig\|^2 = \|F+iG\|^2$
$\|f-ig\|^2 = \|F-iG\|^2$ )

2. Let. $$\tilde{f}(x) = \int_{-\infty}^{\infty} e^{2\pi i x s} F(s)\, ds.$$

Want: $\tilde{f} = f$.

Equivalently: $$\left[\int_{-\infty}^{\infty} \tilde{f}(x)\cdot \overline{g(x)}\, dx = \int_{-\infty}^{\infty} f(x)\cdot \overline{g(x)}\, dx, \text{ for every } g.\right]$$

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{2\pi i x s} F(s) ds\right)\cdot \overline{g(x)}\, dx = \int_{-\infty}^{\infty} F(s)\left(\int_{-\infty}^{\infty} \overline{e^{-2\pi i x s}}\, \overline{g(x)}\, dx\right) ds =$$

$$= \int_{-\infty}^{\infty} F(s)\cdot \overline{G(s)}\, ds = \int_{-\infty}^{\infty} f(x)\cdot \overline{g(x)}\, dx.$$

Examples:

1. Recall : $f = \Pi \longrightarrow F = \text{sinc}.$

By inversion formula:

$$g = \text{sinc} \longrightarrow G(s) = \Pi(-s) = \Pi(s).$$

**Rule**: Inversion Formula Rule. (Synthesis Formula Rule).
(Inversion Rule)

| Function | | Fourier Transform. |
|---|---|---|
| If. $f$ | $\xrightarrow{F.T}$ | $F$ |
| Then $g(x) = F(x)$ | $\xrightarrow{F.T.}$ | $G(s) = f(-s).$ |

why:

$$G(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} g(x)\,dx = \int_{-\infty}^{\infty} e^{-2\pi i x s} F(x)\,dx \overset{\text{Inversion.}}{=} f(-s).$$

2. $f(x) = \dfrac{2}{1+4\pi^2 x^2} \longrightarrow F = ?$

Recall: $f_1(x) = e^{-|x|} \dashrightarrow F_1(S) = \dfrac{2}{1+4\pi^2 s^2}$

$f(x) = F_1(x) \xrightarrow[\text{Inversion Rule.}]{\text{By the}} F(s) = f_1(-s) = e^{-|-s|} =$

$= e^{-|s|}.$

# The Unit Gaussian

Let $\gamma(x) = e^{-\pi x^2}$, $\gamma : \mathbb{R} \to \mathbb{R}$

**Want:** $\hat{\gamma}(s) = \Gamma(s)$, its Fourier transform.

**Note.** $\gamma(-x) = \gamma(x)$ and $\gamma$ is real-valued. $\Rightarrow$. $\Gamma$ is also real-valued and symmetric (even).

$$\Gamma(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s}\, e^{-\pi x^2}\, dx = \int_{-\infty}^{\infty} \cos(2\pi x s)\, e^{-\pi x^2} dx - i \underbrace{\int_{-\infty}^{\infty} \sin(2\pi x s) e^{-\pi x^2} dx}_{=0:}$$

because: $x \mapsto \sin(2\pi x s) \cdot e^{-\pi x^2}$ is odd (antisymmetric) and $\int_{-R}^{R} \text{Antisymmetric}(x)\,dx = 0$

$$= \int_{-\infty}^{\infty} \cos(2\pi x s)\, e^{-\pi x^2} dx$$

—.

$$\frac{d\Gamma}{ds} = \frac{d}{ds} \int_{-\infty}^{\infty} \cos(2\pi x s)\, e^{-\pi x^2} dx =$$

$$= \int_{-\infty}^{\infty} \left(\frac{d}{ds}\left(\cos(2\pi x s)\right)\right) \cdot e^{-\pi x^2} dx = -2\pi \int_{-\infty}^{\infty} \sin(2\pi x s) \cdot x\, e^{-\pi x^2} dx =$$

$$\frac{d}{dx}\left(e^{-\pi x^2}\right) = -2\pi x \cdot e^{-\pi x^2}$$

$$= \int_{-\infty}^{\infty} \sin(2\pi x s) \cdot \frac{d}{dx}\left(e^{-\pi x^2}\right) dx \overset{\text{integration by parts}}{=}$$

$$= \sin(2\pi x s) \cdot e^{-\pi x^2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 2\pi s \cdot \cos(2\pi x s)\, e^{-\pi x^2} dx =$$

$$= 0 - 2\pi s \int_{-\infty}^{\infty} \cos(2\pi x s)\, e^{-\pi x^2} dx = -2\pi s \cdot \Gamma(s)$$

Thus:

$$\frac{d\Gamma}{ds} + 2\pi s \cdot \Gamma(s) = 0.$$

$$\frac{d\Gamma}{\Gamma} = -2\pi s \cdot ds$$

$$\log|\Gamma| = -\pi s^2 + C \Rightarrow \Gamma(s) = \Gamma(0) \cdot e^{-\pi s^2}$$

$$\Gamma(0) = \int_{-\infty}^{\infty} e^{-2\pi i x \cdot 0} \cdot e^{-\pi x^2} dx = \int_{-\infty}^{\infty} e^{-\pi x^2} dx \overset{\text{see HW 2.}}{=} 1.$$

<u>Conclusion</u>: $\Gamma(s) = e^{-\pi s^2}$

$$\gamma(x) = e^{-\pi x^2} \xrightarrow{\text{F.T.}} \Gamma(s) = e^{-\pi s^2} = \gamma(s).$$

<u>Examples</u>:

1. $f_1(x) = e^{-x^2}, \longrightarrow F_1 = ?$

2. $f_2(x) = \cos(2x) \cdot e^{-x^2} \longrightarrow F_2 = ?$

<u>Solution</u>.

| <u>Function</u> | <u>Fourier Transform</u> |
|---|---|
| $\gamma(x) = e^{-\pi x^2}$ | $\hat{\gamma}(s) = e^{-\pi s^2}$ |

$$f_1(x) = \gamma\left(\frac{x}{\sqrt{\pi}}\right) \xrightarrow[\text{Scaling Rule.}]{\text{by}} F_1(s) = \sqrt{\pi}\,\hat{\gamma}(\sqrt{\pi} s) = \sqrt{\pi}\, e^{-\pi^2 s^2}$$

$$\gamma\left(\frac{x}{\sqrt{\pi}}\right) = e^{-\pi\left(\frac{x}{\sqrt{\pi}}\right)^2} = e^{-\cancel{\pi}\frac{x^2}{\cancel{\pi}}} = e^{-x^2} \quad \Big| \; a = \frac{1}{\sqrt{\pi}}$$

$$f_2(x) = \frac{1}{2}\left(e^{i2x} + e^{-i2x}\right) e^{-x^2} = \frac{1}{2} e^{i2x} f_1(x) + \frac{1}{2} e^{-i2x} \cdot f_1(x) \xrightarrow[\text{Rule}]{\text{Modulation}}$$

$$f_2(x) = \frac{1}{2}e^{2ix} f_1(x) + \frac{1}{2}e^{-2ix} f_1(x) \longrightarrow F_2(s) = \frac{1}{2}F_1\left(s-\frac{1}{\pi}\right) + \frac{1}{2}F_1\left(s+\frac{1}{\pi}\right) =$$

$$= \frac{\sqrt{\pi}}{2} e^{-\pi^2 (s-\frac{1}{\pi})^2} + \frac{\sqrt{\pi}}{2} e^{-\pi^2 (s+\frac{1}{\pi})^2}$$

We obtained:

$$\cos(2x)\, e^{-x^2} \xrightarrow{F.T.} \frac{\sqrt{\pi}}{2} e^{-\pi^2(s-\frac{1}{\pi})^2} + \frac{\sqrt{\pi}}{2} e^{-\pi^2(s+\frac{1}{\pi})^2}.$$

---

<u>Rules</u>: Derivative Rule

Power Scaling Rule.

Assume $f$ satisfies "some" regularity.

| | <u>Function.</u> | <u>Fourier Transform</u> |
|---|---|---|
| If: | $f$ | $F$ |
| let | $g(x) = f'(x)$ | $G(s) = 2\pi i s \cdot F(s)$ |
| | $h(x) = x \cdot f(x)$ | $H(s) = -\frac{1}{2\pi i} F'(s) = \frac{i}{2\pi}\frac{d}{ds}(F(s))$ |

---

<u>why</u>:

$$G(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} g(x)\,dx = \int_{-\infty}^{\infty} e^{-2\pi i x s} f'(x)\,dx \overset{\text{Integration by parts.}}{=}$$

$$= \underbrace{e^{-2\pi i x s}\cdot f(x)\Big|_{-\infty}^{\infty}}_{\overset{\shortparallel}{0}} - \int_{-\infty}^{\infty} (-2\pi i s)\, e^{-2\pi i x s} f(x)\,dx = 2\pi i s \int_{-\infty}^{\infty} e^{-2\pi i x s} f(x)\,dx =$$

$$= 2\pi i s \cdot F(s).$$

"some" conditions: $\lim_{x\to\pm\infty} f(x) = 0$.

$H(s) = \dots$ (similar). / or use inversion formula.

**Theorem** (see page 153)

Assume: (1) $f$ is in $C^{(n-1)}(\mathbb{R})$, $n-1$ times differentiable with continuous $f^{(n-1)}$.

and $f$ is piecewise $C^{(n)}(\mathbb{R})$, with bounded $n$th derivative.

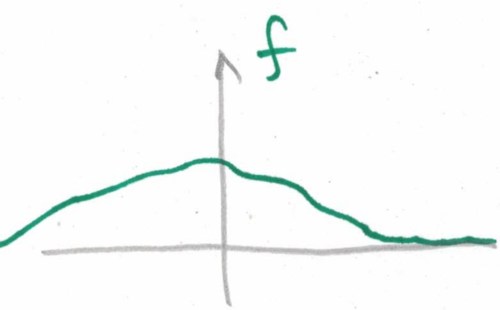
(2). $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$, $\int_{-\infty}^{\infty} |f'(x)|\,dx < \infty$, ..., $\int_{-\infty}^{\infty} |f^{(n)}(x)|\,dx < \infty$

**Then**:

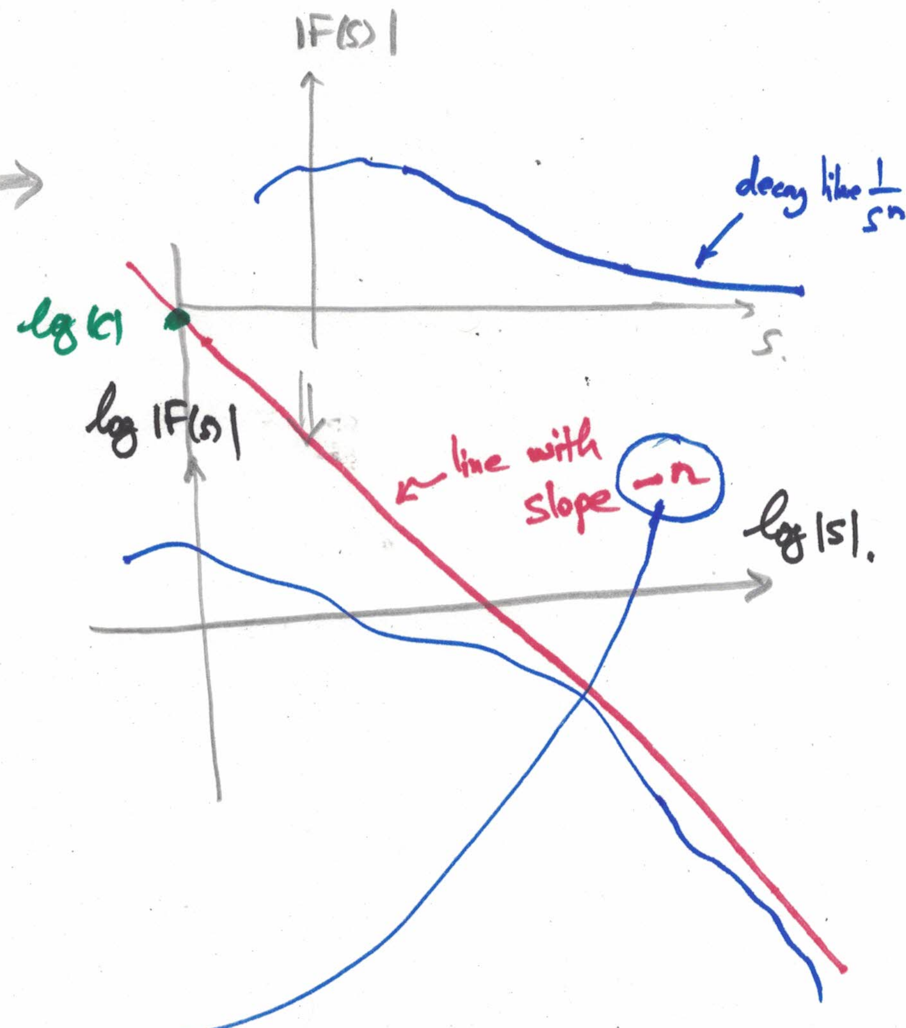
$$|F(s)| \leq \frac{C}{(1+|s|)^n}, \text{ for some } C>0.$$

$\longrightarrow$ Fourier transform of $f$ decays like $\frac{1}{s^n}$ (as $s \to \infty$)

---




$\Rightarrow$




$$\log|F(s)| \leq \log\frac{C}{|s|^n} =$$

$$= \log(c) - n \log|s|.$$

# Poisson Summation Formula

Start: $f : \mathbb{R} \to \mathbb{C}$

Step 1. Construct its 1-periodization:

$$g:\mathbb{R}\to\mathbb{C},\qquad g(x)=\sum_{n=-\infty}^{\infty} f(x-n)$$

$$\left[\ g(x+1)=\sum_{n=-\infty}^{\infty} f(x+1-n) \underset{m=n-1}{=} \sum_{m=-\infty}^{\infty} f(x-m)=g(x).\ \right]$$

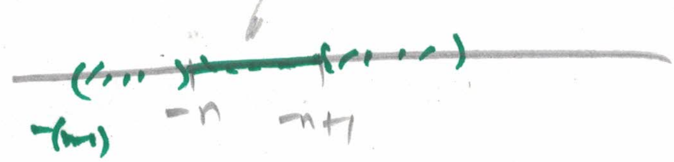
Step 2.

$$g(x)=\sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x},\qquad c_k=\int_0^1 e^{-2\pi i k x}\, g(x)\,dx$$

$$c_n=\int_0^1 e^{-2\pi i k x}\sum_{n=-\infty}^{\infty} f(x-n)\,dx=\sum_{n=-\infty}^{\infty}\int_0^1 e^{-2\pi i k x} f(x-n)\,dx=$$

$y=x-n.$

$$=\sum_{n=-\infty}^{\infty}\int_{-n}^{-n+1}\underbrace{e^{-2\pi i k(y+n)}}_{e^{-2\pi i k y}\cdot\underbrace{e^{-2\pi i k n}}_{1}} f(y)\,dy. = \sum_{n=-\infty}^{\infty}\underbrace{\int_{-n}^{-n+1} e^{-2\pi i k y} f(y)\,dy}. =$$




$$=\int_{-\infty}^{\infty} e^{-2\pi i k y} f(y)\,dy = F(k).$$

Step 3

$$\boxed{\sum_{n=-\infty}^{\infty} f(x-n) = \underbrace{\sum_{k=-\infty}^{\infty} F(k) e^{2\pi i k x}}_{\text{Poisson Summation Formula}}}$$