L11

Today: Convolutions, Integral Equations, Sine and Cosine Transforms

Functions defined over $\mathbb{R}$:

Def. Let $f, g : \mathbb{R} \longrightarrow \mathbb{C}$.
The convolution of f and g, denoted $f * g$ is the function $f*g : \mathbb{R} \longrightarrow \mathbb{C}$,

$$f*g\,(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy.$$

Periodic Functions:

Def. Let $f, g : \mathbb{R} \to \mathbb{C}$, 1-periodic functions: $f(x+1) = f(x)$, $g(x+1) = g(x)$.

The (periodic) convolution of f and g, denoted $f*g$ is the 1-periodic function, $f*g : \mathbb{R} \to \mathbb{C}$,

$$f*g\,(x) = \int_0^1 f(x-y)\, g(y)\, dy$$

Sequences:

Definition Let $f, g : \mathbb{Z} \longrightarrow \mathbb{C}$ be two sequences indexed by positive and negative integers,

$$f = (f_n)_{n \in \mathbb{Z}}, \quad g = (g_n)_{n \in \mathbb{Z}}.$$

The convolution of sequences f and g, denoted $f * g$ is the sequence

$$(f*g)_n = \sum_{k=-\infty}^{\infty} f_{n-k} \cdot g_k \quad , \text{ for every } n \in \mathbb{Z}.$$

(Finite-dimensional) Vectors:

Definition Let $f, g \in \mathbb{C}^N$, $N \geq 1$ integer.

The convolution of f and g, denoted $f * g$ is the N-vector:

$$(f*g)_{n+1} = \sum_{k=0}^{N-1} f_{(n-k) \bmod N + 1} \cdot g_{k+1} \quad , \quad 0 \leq n \leq N-1$$

Remark If we index vectors from 0 to $N-1$, then:

$$(f*g)_n = \sum_{k=0}^{N-1} f_{(n-k) \bmod N} \cdot g_k \quad , \quad 0 \leq n \leq N-1.$$

Today we focus on:

$$f, g : \mathbb{R} \longrightarrow \mathbb{C}$$

$$f * g\,(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy$$

Example. Assume $f(x) = g(x) = e^{-x^2}$.

Find $f * g = ?$

Solution.

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy = \int_{-\infty}^{\infty} e^{-(x-y)^2} \cdot e^{-y^2}\, dy =$$

$$= \int_{-\infty}^{\infty} e^{-(x^2 - 2xy + y^2) - y^2}\, dy = \int_{-\infty}^{\infty} e^{-x^2 + 2xy - 2y^2}\, dy =$$

$$= \int_{-\infty}^{\infty} \exp\left(-2\left(y^2 - xy + \frac{x^2}{4}\right) - \frac{x^2}{2}\right) dy = \int_{-\infty}^{\infty} \exp\left(-2\left(y - \frac{x}{2}\right)^2\right) \cdot \underbrace{\exp\left(-\frac{x^2}{2}\right)}\, dy$$

$$= e^{-\frac{x^2}{2}} \cdot \int_{-\infty}^{\infty} e^{-2\left(y - \frac{x}{2}\right)^2}\, dy = e^{-\frac{x^2}{2}} \underbrace{\int_{-\infty}^{\infty} e^{-2s^2}\, ds}_{I}.$$

$$s = y - \frac{x}{2}$$
$$ds = dy$$

$I = ?$

Unit Gaussian:

$$\gamma(x) = e^{-\pi x^2} \xrightarrow{F.T.} \Gamma(s) = e^{-\pi s^2}$$

$$\int_{-\infty}^{\infty} e^{-2\pi i s x} \cdot e^{-\pi x^2} \, dx = e^{-\pi s^2}$$

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = e^{-0} = 1.$$

$$\sqrt{\pi}\, x = \sqrt{2}\, s \longrightarrow x = \sqrt{\frac{2}{\pi}}\, s.$$

$$dx = \sqrt{\frac{2}{\pi}}\, ds$$

$$\int_{-\infty}^{\infty} e^{-2s^2} \sqrt{\frac{2}{\pi}}\, ds = 1.$$

$$\underbrace{\int_{-\infty}^{\infty} e^{-2s^2} ds = \sqrt{\frac{\pi}{2}} = I}$$

$$\rightarrow \boxed{f * g\,(x) = \sqrt{\frac{\pi}{2}}\, e^{-\frac{x^2}{2}}}$$

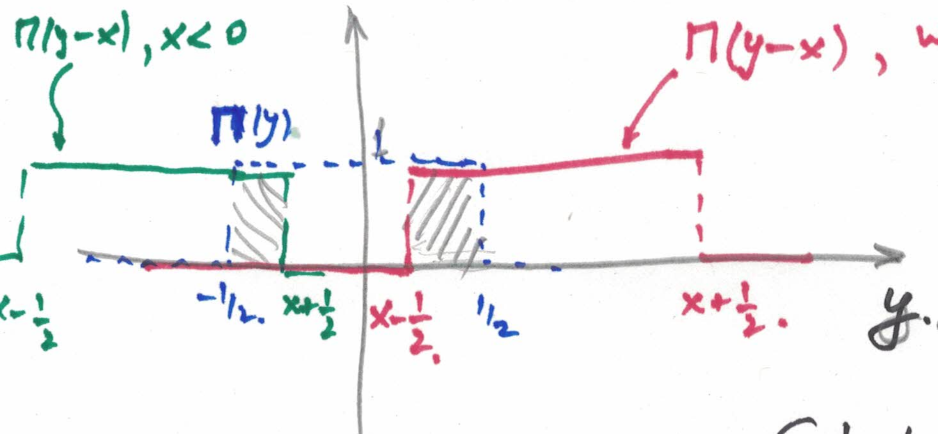
<u>Example</u>. $f = g = \Pi$, $\Pi(x) = \begin{cases} 1, & -\frac{1}{2} < x < \frac{1}{2} \\ 0, & |x| > \frac{1}{2} \end{cases}$

<u>Problem</u>: $f * g = ?$

(5).

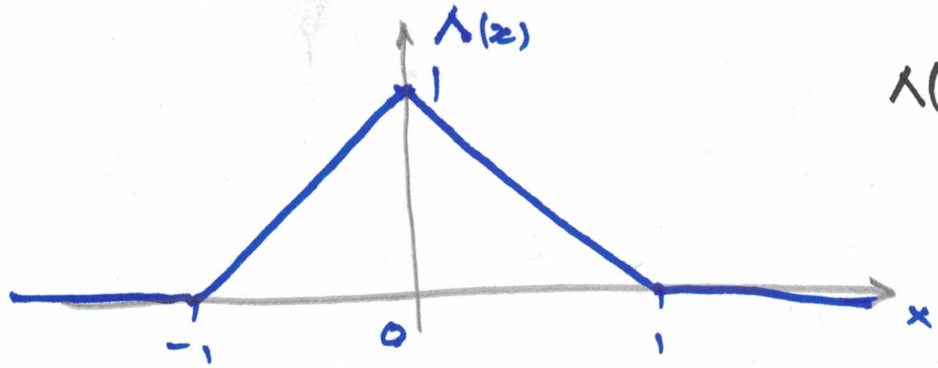
$$\Pi * \Pi(x) = \int_{-\infty}^{\infty} \Pi(x-y)\cdot\Pi(y)\,dy = \int_{-1/2}^{1/2} \Pi(x-y)\,dy. =$$

$$= \int_{-1/2}^{1/2} \Pi(y-x)\,dy$$



$$\Pi * \Pi(x) = \int_{-1/2}^{1/2} \Pi(y-x)\,dy = \begin{cases} \frac{1}{2}-\left(x-\frac{1}{2}\right) = 1-x, & 0 \le x \le 1. \\ 0, & 1 < x \\ x+\frac{1}{2}-\left(-\frac{1}{2}\right) = x+1 = 1+x, & -1 \le x < 0. \\ 0, & x < -1 \end{cases} =$$

$$= \begin{cases} 1-|x|, & -1 \le x \le 1. \\ 0, & \text{otherwise}. \end{cases} =: \Lambda(x).$$



$$\Lambda(x) = \begin{cases} 1-|x|, & |x| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

<u>Theorem</u>. Under some regularity/ integrability conditions:

	<u>Function.</u>		<u>Fourier Transform</u>
If.	f	$\longrightarrow$	F
	g	$\longrightarrow$	$G.$
Then:	$h = f * g$	$\longrightarrow$	$H = F \cdot G.$
	$r = f \cdot g$	$\longrightarrow$	$R = F * G.$

<u>Why</u>?

$$f * g \longrightarrow F \cdot G.$$

Let $h = f * g$.

$$H(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} h(x)\,dx = \int_{-\infty}^{\infty} e^{-2\pi i x s} \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy\,dx =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{e^{-2\pi i x s}} f(x-y)\,g(y)\,dx\,dy =$$

$$e^{-2\pi i (x-y+y)s} = e^{-2\pi i (x-y)\cdot s - 2\pi i y \cdot s} =$$

$$= e^{-2\pi i (x-y)\cdot s} \cdot e^{-2\pi i y s}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i (x-y)s} f(x-y) \cdot e^{-2\pi i y s} g(y)\,dx\,dy. =$$

$$= \int_{-\infty}^{\infty} e^{-2\pi i y s} g(y) \cdot \left(\int_{-\infty}^{\infty} e^{-2\pi i (x-y) s} f(x-y)\, dx \right) dy =$$

$u = x - y.$

$du = dx$

$$= \int_{-\infty}^{\infty} e^{-2\pi i y s} g(y) \underbrace{\left(\int_{-\infty}^{\infty} e^{-2\pi i u s} f(u)\, du \right)}_{F(s)} dy = F(s) \cdot \underbrace{\int_{-\infty}^{\infty} e^{-2\pi i y s} g(y)\, dy}_{G(s)} = F(s) \cdot G$$

$y \mapsto f(x-y).$ $\qquad \Vert \qquad y \mapsto f(y).$

$y \mapsto f(-y) = \tilde{f}(y).$

$y \mapsto \tilde{f}(y-x) = f(-(y-x)) = \underline{\underline{f(x-y)}}.$

<u>Property</u>: $f * g = g * f.$

<u>why</u>:

$$g * f\,(x) = \int_{-\infty}^{\infty} g(x-y)\, f(y)\, dy = \int_{-\infty}^{\infty} g(t) \cdot f(x-t)\, dt =$$

$t = x - y.$

$$= \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt = (f * g)(x).$$

Revisit example: $f(x) = g(x) = e^{-x^2}$

Compute F, G, then $H = F \cdot G$, then $h = \mathcal{F}^{-1}(H)$.

Sine and Cosine Transform.

Let. $f : [0,\infty) \to \mathbb{C}$.

half of $\mathbb{R}$.

The cosine transform $F_c : [0,\infty) \to \mathbb{C}$ is given by:

$$F_c(s) = 2\int_0^\infty f(x)\cos(2\pi xs)\,dx.$$

The sine transform $F_s : [0,\infty) \to \mathbb{C}$ is given by:

$$F_s(s) = 2\int_0^\infty f(x)\sin(2\pi xs)\,dx.$$

Extend the definitions of F_c and F_s to $\mathbb{R}$:

$$F_c(s) = 2\int_0^\infty f(x)\cos(2\pi xs)\,dx, \text{ for every } s\in\mathbb{R} : \underline{F_c(-s) = F_c(s)}.$$

$$F_s(s) = 2\int_0^\infty f(x)\sin(2\pi xs)\,dx, \text{ for every } s\in\mathbb{R} : \underline{F_s(-s) = -F_s(s)}.$$

Inversion Formulas:

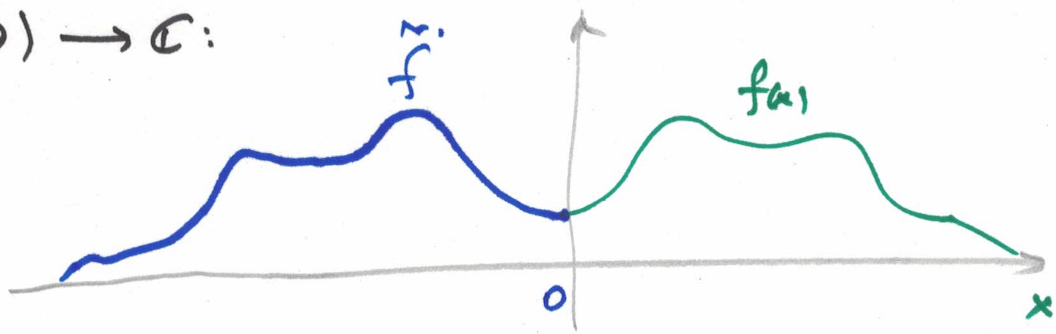
Given $F_c, F_s : [0,\infty) \longrightarrow \mathbb{R}$, for every $x \geq 0$:

$$f(x) = 2\int_0^\infty F_c(s)\cdot\cos(2\pi x s)\,ds \quad , \quad x \geq 0$$

$$f(x) = 2\int_0^\infty F_s(s)\,\sin(2\pi x s)\,ds \quad , \quad x \geq 0.$$

Why:

Start $f : [0,\infty) \longrightarrow \mathbb{C}$:



Construct $\tilde{f} : \mathbb{R} \to \mathbb{C}$,
$$\tilde{f}(x) = \begin{cases} f(x), & x \geq 0. \\ f(-x), & x \leq 0 \end{cases}$$

Compute Fourier transform of $\tilde{f}$, say $\tilde{F}$:

$$\tilde{F}(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s}\,\tilde{f}(x)\,dx = \int_{-\infty}^{0} e^{-2\pi i x s}\,\tilde{f}(x)\,dx + \int_0^\infty e^{-2\pi i x s}\,\tilde{f}(x)\,dx$$

$t = -x \qquad\qquad t = x$

$$= \int_0^\infty e^{2\pi i t s}\,\underbrace{\tilde{f}(-t)}_{\tilde{f}(t) = f(t)}\,dt + \int_0^\infty e^{-2\pi i t s}\,\underbrace{\tilde{f}(t)}_{= f(t)}\,dt =$$

$$\tilde{F}(s) = \int_0^\infty \underbrace{\left(e^{2\pi i t s} + e^{-2\pi i t s}\right)}_{2\cos(2\pi t s).} f(t)\,dt \underset{t \to x}{=} 2\int_0^\infty \cos(2\pi x s)\, f(x)\,dx = F_c(s).$$

<u>Conclusion:</u> The cosine transform is the Fourier transform of the even extension of f.

How to invert:

$$F_c = \tilde{F} \xrightarrow[\text{inversion.}]{\text{F.T.}} \tilde{f}(x) = \int_{-\infty}^{\infty} e^{2\pi i x s}\, \tilde{F}(s)\,ds =$$

$$= \int_{-\infty}^{0} e^{2\pi i x s}\, \tilde{F}(s)\,ds + \int_0^\infty e^{2\pi i x s}\, \tilde{F}(s)\,ds =$$

$$= \int_0^\infty e^{-2\pi i x s}\, \tilde{F}(-s)\,ds + \int_0^\infty e^{2\pi i x s}\, \tilde{F}(s)\,ds =$$

But $\tilde{F}(-s) = F_c(-s) = F_c(s)$, $\quad \tilde{F}(s) = F_c(s)$.

$$= \int_0^\infty \underbrace{\left(e^{-2\pi i x s} + e^{2\pi i x s}\right)}_{2\cos(2\pi x s)} F_c(s)\,ds = 2\int_0^\infty F_c(s)\cos(2\pi x s)\,ds$$

$$\Rightarrow \quad f(x) = 2\int_0^\infty F_c(s)\cos(2\pi x s)\,ds, \quad \text{for } x \geq 0.$$

Similarly for the sine transform $\longrightarrow$ Use the odd (antisymmetric