L13

Sampling of Bandlimited Functions

Problems:

1) Concept of bandlimited functions; examples

2) Shannon's Sampling Formula: Consequence of an ONB for the space of bandlimited functions. Refinements.

3) Approximation Errors for bandlimited functions and (more) general functions.

I Bandlimited functions.

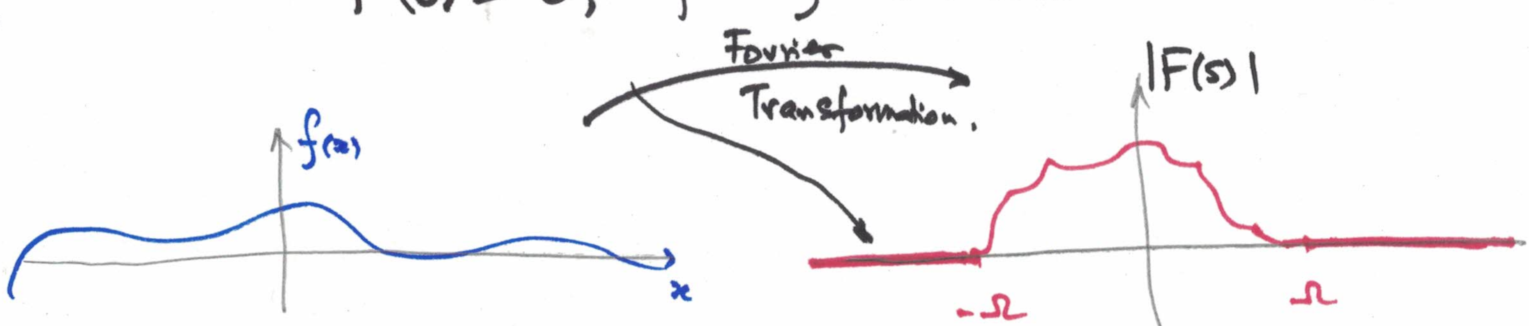
Recall: $f : \mathbb{R} \longrightarrow \mathbb{C}$, its Fourier transform $F : \mathbb{R} \to \mathbb{C}$,

$$F(s) = \int_{-\infty}^{\infty} e^{-2\pi i s x} f(x)\, dx.$$

Recall the inversion formula: $$f(x) = \int_{-\infty}^{\infty} e^{2\pi i s x} F(s)\, ds.$$

Definition A function $f : \mathbb{R} \to \mathbb{C}$ is called Ω-bandlimited if its Fourier transform F vanishes outside the interval $[-\Omega, \Omega]$ i.e.,

$$F(s) = 0, \quad \text{for any } s < -\Omega \text{ or } s > \Omega.$$

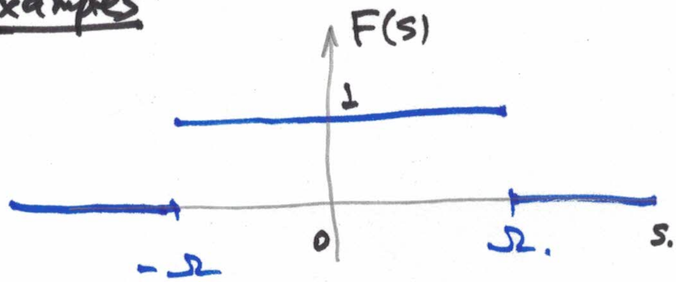


We shall focus on the space:

$$B^2_\Omega = \Big\{ f : \mathbb{R} \to \mathbb{C} \;:\; \underbrace{\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty}_{f \in L^2(\mathbb{R})} \;\; \underline{\text{and}} \;\; \underbrace{F(s)=0, \text{ for } |s|>\Omega}_{f \text{ is } \Omega\text{-bandlimited.}} \Big\}$$

__Examples__

1).



Construct / Choose:

$$F(s) = \begin{cases} 1, & -\Omega < s < \Omega \\ 0, & |s| > \Omega. \end{cases}$$

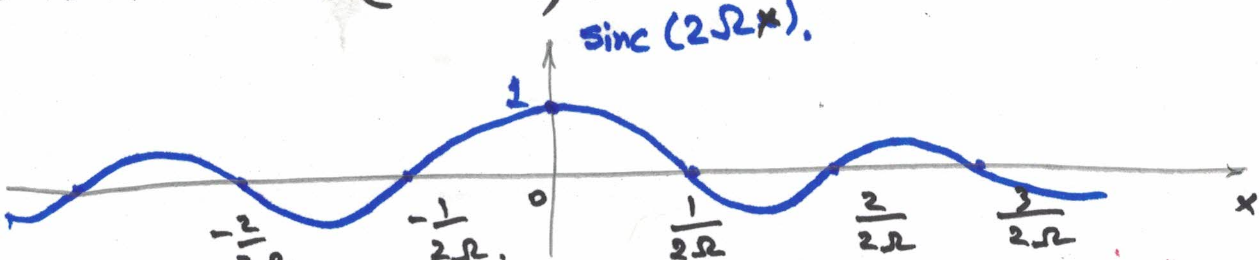
Compute f:

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i s x} F(s)\, ds = \int_{-\Omega}^{\Omega} e^{2\pi i s x}\, ds \underset{\text{for } x \neq 0}{=} \frac{e^{2\pi i \Omega x} - e^{-2\pi i \Omega x}}{2\pi i x} =$$

$$= \frac{\cancel{2i} \sin(2\pi\Omega x)}{2\pi \cancel{i} x} = \frac{\sin(2\pi\Omega x)}{2\Omega\pi x} \cdot 2\Omega = 2\Omega \cdot \operatorname{sinc}(2\Omega x)$$

For $x=0$: $f(0) = \int_{-\Omega}^{\Omega} 1\, dx = 2\Omega$

$$f(x) = 2\Omega \cdot \operatorname{sinc}(2\Omega x).$$



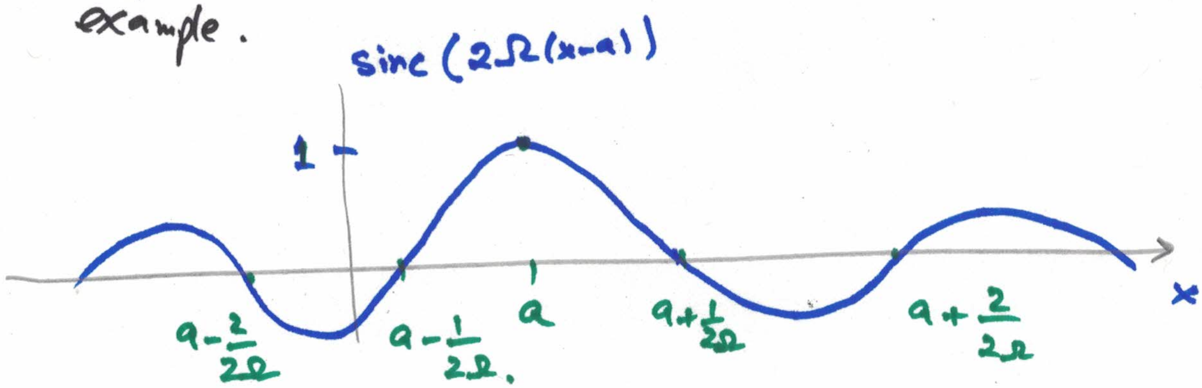
2). Choose/Construct:

Fix $a \in \mathbb{R}$,
$$F_a(s) = \begin{cases} e^{-2\pi i s a}, & |s| < \Omega. \\ 0, & |s| > \Omega. \end{cases}$$

Note: $F_a(s) = e^{-2\pi i s a} \cdot \underbrace{F(s)}_{\text{from previous example.}}$

$$f_a(x) = \int_{-\infty}^{\infty} e^{2\pi i s x} \cdot e^{-2\pi i s a} F(s)\,ds = \int_{-\infty}^{\infty} e^{2\pi i s(x-a)} \cdot F(s)\,ds =$$

$$= \underbrace{f(x-a)}_{\text{from previous example.}} = 2\Omega \cdot \operatorname{sinc}\big(2\Omega(x-a)\big)$$



Example of an Ω-bandlimited function.

<u>Remark:</u>

If f is ~~any~~ Ω-bandlimited function then any shift (translate) is also an Ω-bandlimited function.

B^2_Ω is a shift-invariant space.

$\boxed{\text{II}}$ Shannon's Sampling Formula.

$$B^2_{\Omega} \xleftarrow{\ \mathcal{F}^{-1}\ } L^2[-\Omega, \Omega].$$

$\uparrow$ ONB (under B^2_Ω) $\longleftarrow$ Pull-back through the inverse Fourier transform. $\longleftarrow$ ONB $\uparrow$ (under $L^2[-\Omega,\Omega]$)

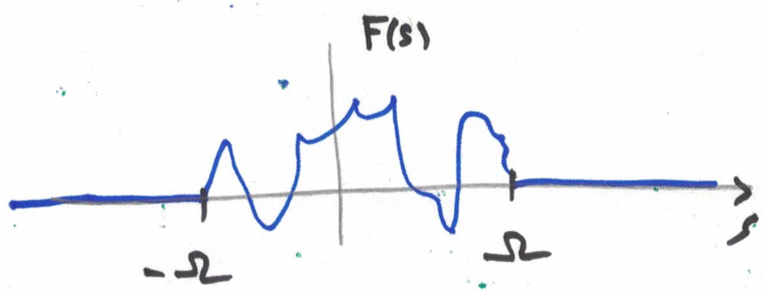
For $L^2[-\Omega, \Omega\}$ $\longrightarrow$ ONB : $E_n(s) = \begin{cases} \frac{1}{\sqrt{2\Omega}} e^{-2\pi i \frac{ns}{2\Omega}}, & |s| < \Omega \\ 0, & |s| > \Omega. \end{cases}$

$n \in \mathbb{Z}$

$\{ \ldots, E_{-2}, E_{-1}, E_0, E_1, E_2, \ldots \}$ is ONB for:

$$\{ F \in L^2(\mathbb{R}) : \quad F(s) = 0, \ |s| > \Omega \}$$

Why: Because of the Fourier Series Expansion.



The ONB in B^2_Ω : $e_n = \mathcal{F}^{-1}(E_n)$, n integer.

$$e_n(x) = \int_{-\infty}^{\infty} e^{2\pi i x s} E_n(s)\, ds = \frac{1}{\sqrt{2\Omega}} \int_{-\Omega}^{\Omega} e^{2\pi i x s - 2\pi i \frac{n}{2\Omega} s} ds =$$

$$= \frac{1}{\sqrt{2\Omega}} \int_{-\Omega}^{\Omega} e^{2\pi i s (x - \frac{n}{2\Omega})} ds = \frac{1}{\sqrt{2\Omega}} \; \frac{e^{2\pi i \Omega (x - \frac{n}{2\Omega})} - e^{-2\pi i \Omega (x - \frac{n}{2\Omega})}}{2\pi i (x - \frac{n}{2\Omega})} =$$

For. $x - \frac{n}{2\Omega} \neq 0$

$$= \frac{2\Omega}{\sqrt{2\Omega}} \frac{\cancel{2i} \sin\left(2\pi\Omega\left(x-\frac{n}{2\Omega}\right)\right)}{\cancel{2i}\,\pi\left(x-\frac{n}{2\Omega}\right)2\Omega} = \sqrt{2\Omega}\ \operatorname{sinc}(2\Omega x - n)$$

Therefore, we obtained:

$$\boxed{\begin{array}{l}\left\{e_n \,;\ e_n(x) = \sqrt{2\Omega}\ \operatorname{sinc}(2\Omega x - n),\ n \in \mathbb{Z}\right\} \\ \text{is an Orthonormal Basis (ONB) for } B^2_\Omega .\end{array}}$$

Why is this important:

Take any $f \in B^2_\Omega$,

$$\sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n = f.$$

is the expansion of f w.r.t. this ONB.

Q: What are $\langle f, e_n \rangle = ?$ Use Plancherel/Parseval Identity

A:

$$\langle f, e_n \rangle = \int_{-\infty}^{\infty} f(x) \cdot \overline{e_n(x)}\, dx = \int_{-\infty}^{\infty} F(s) \cdot \overline{E_n(s)}\, ds =$$

$$= \int_{-\Omega}^{\Omega} F(s) \cdot \frac{1}{\sqrt{2\Omega}} e^{+2\pi i \frac{n}{2\Omega} s}\, ds = \int_{-\infty}^{\infty} F(s) \frac{1}{\sqrt{2\Omega}} e^{2\pi i \frac{n}{2\Omega} s}\, ds =$$

f is in B^2_Ω

$$= \frac{1}{\sqrt{2\Omega}} \int_{-\infty}^{\infty} e^{2\pi i \frac{n}{2\Omega} s} F(s)\, ds = \frac{1}{\sqrt{2\Omega}} f\left(\frac{n}{2\Omega}\right).$$

Thus we obtained:

$$f(x) = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n(x) = \sum_{n=-\infty}^{\infty} \frac{1}{\cancel{\sqrt{2\Omega}}} f\left(\frac{n}{2\Omega}\right) \cancel{\sqrt{2\Omega}} \operatorname{sinc}(2\Omega x - n)$$

Borel - Whittaker - Kotelnikov - Shannon Formula:

$$f \in B^2_\Omega, \qquad f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2\Omega}\right) \cdot \operatorname{sinc}(2\Omega x - n)$$

Notation: $T_c = \dfrac{1}{2\Omega}$: critical sampling period.

$$f(x) = \sum_{n=-\infty}^{\infty} f(n \cdot T_c) \cdot \operatorname{sinc}\left(\frac{x - n \cdot T_c}{T_c}\right).$$

Thus: If we measure / know / are given

$$\{ f(n \cdot T_c), \ n \in \mathbb{Z} \}.$$

then we can can compute $f(x)$ for any $x \in \mathbb{R}$.

If:

$$\{ \ldots, f(-100 \cdot T_c), f(-99 \cdot T_c), \ldots, f(0), f(T_c), \ldots, f(1000 T_c), \ldots \}$$

are known then we can compute $f(0.21 \cdot T_c)$, or ~~any~~ for any other x

Notation: $\dfrac{1}{T_c} = 2\Omega$: Nyquist sampling rate.

$\longrightarrow$ we need to acquire 2Ω samples for each unit of time.

Refinement 1

<u>Observation</u>: If f is an Ω- bandlimited function, then f is also an Ω'-bandlimited function, for any $\Omega' > \Omega$

$$B^2_{\Omega} \subset B^2_{\Omega'}, \quad \text{for } \Omega' > \Omega.$$

Thus, reconstruction should also work using Ω' instead of Ω:

$$f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2\Omega'}\right) \cdot \operatorname{sinc}\left(2\Omega' x - n\right).$$

, for any $\Omega' \geq \Omega$.

$T = \frac{1}{2\Omega'}$ represents the <u>sampling period</u>.

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} f(n \cdot T) \cdot \operatorname{Sinc}\left(\frac{x - nT}{T}\right)}$$

where $T \leq T_c$

when $T < T_c$ $(\Omega' > \Omega)$ $\longrightarrow$ we say <u>oversampling</u>.

Application: Assume $f: \mathbb{R} \to \mathbb{R}$ is a 1 KHz-bandlimited signal.

Question 1: What is the maximum sampling period T_c so that $f(x)$ can be reconstructed from its samples $\{f(n \cdot T_c)\}, n \in \mathbb{Z}$

Answer:

$$T_c = \frac{1}{2\Omega} = \frac{1}{2 \cdot 10^3 \frac{1}{s}} = \frac{1}{2} 10^{-3} s = 0.5 \text{ ms}$$

Nyquist rate: $\frac{1}{T_c} = 2000 \text{ Hz} = 2000$ samples/s.

If f is 1 MHz-bandlimited $\dashrightarrow T_c = \frac{1}{2 \cdot 10^6 \frac{1}{s}} = 0.5 \mu s$

Nyquist rate: $2 \text{ MHz} = 2{,}000{,}000$ samples/s.

Question 2: If f is 1 MHz-bandlimited and if we sample with a sampling period $T = 0.1 \mu s$, can we compute $f(x)$ from its samples?

Answer:

Since $T = 0.1 \mu s < 0.5 \mu s = T_c \Rightarrow$ Answer is YES.