(L16) 

# Test Functions

**Theorem** [Properties of the class of Schwartz test functions, $\mathcal{S}$].

Let $\phi, \phi_1, \phi_2, \psi \in \mathcal{S}$.

1). For any $a_1, a_2 \in \mathbb{C}$, $a_1\phi_1 + a_2\phi_2 \in \mathcal{S}$

2) For any $x_0 \in \mathbb{R}$, $\phi_{x_0}(x) = \phi(x - x_0)$, then $\phi_{x_0} \in \mathcal{S}$.

3) For any $a > 0$, $\phi_a(x) = \phi(a \cdot x)$, then $\phi_a \in \mathcal{S}$.

4) For any $w_0 \in \mathbb{R}$, $\phi_{w_0}(x) = e^{2\pi i w_0 \cdot x}\phi(x)$, then $\phi_{w_0} \in \mathcal{S}$.

5) For any $n \geq 0$, let $\phi_n(x) = x^n \cdot \phi(x)$. Then $\phi_n \in \mathcal{S}$.

6) For any $n \geq 1$, let $\phi^{(n)}(x) = \frac{d^n}{dx^n}\phi(x)$. Then $\phi^{(n)} \in \mathcal{S}$.

7) Let $\psi_1(x) = \overline{\phi(x)}$, $\psi_2(x) = \phi(-x)$. Then $\psi_1, \psi_2 \in \mathcal{S}$.

$\boxed{\text{8). Let } \hat{\phi} \text{ denote the Fourier transform of } \phi. \text{ Then } \hat{\phi} \in \mathcal{S}.}$ !

and: $\hat{\phi}(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s}\phi(x)\,dx$, $\phi(x) = \int_{-\infty}^{\infty} e^{2\pi i x s}\hat{\phi}(s)\,ds$.

both integrals converge absolutely, and pointwise for any $x \in \mathbb{R}$.

9) $\phi_1, \phi_2 \in \mathcal{S} \implies \phi_1 \cdot \phi_2 \in \mathcal{S}$

10) $\phi_1, \phi_2 \in \mathcal{S} \implies \phi_1 * \phi_2 \in \mathcal{S}$ (convolution)

11) Assume $\phi(x_0) = 0$. Let

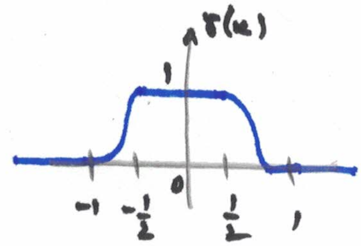
$$\psi(x) = \begin{cases} \dfrac{\phi(x)}{x - x_0}, & x \neq x_0 \\ \phi'(x_0), & x = x_0. \end{cases}$$

Then $\psi \in \mathcal{S}$.

12) Assume $\int_{-\infty}^{\infty} \phi(x)\,dx = 0$. Let $\psi(x) = \int_{-\infty}^{x} \phi(y)\,dy$.

Then $\psi \in \mathcal{S}$.

13) Assume $\gamma : \mathbb{R} \to \mathbb{R}$ is the mesa function:




~~Then~~ Let:

$$\psi(x) = \begin{cases} \dfrac{\phi(x) - \phi(0) \cdot \gamma(x)}{x}, & x \neq 0 \\ \phi'(0), & x = 0. \end{cases}$$

Then $\psi \in \mathcal{S}$.

<u>why</u>:

5) : $\phi_n(x) = x^n \cdot \phi(x)$. $\longrightarrow$ $\phi_n \in \mathcal{S}$.

1) $\phi_n$ is $C^\infty$ because $x \longmapsto x^n$ and $x \longmapsto \phi(x)$ both are $C^\infty$.

2). $\phi_n^{(N)}(x) = c_0 \cdot x^{n-N} \cdot \phi(x) + c_1 \cdot x^{n-N+1} \cdot \phi'(x) +$

the $N^{th}$ derivative

$$+ \dots + c_k \cdot x^{n-N+k} \cdot \phi^{(k)}(x) + \dots + c_N\, x^n \cdot \phi^{(N)}(x).$$

$$(1+|x|)^M \cdot |\phi_n^{(N)}(x)| \leq |c_0| \cdot \overbrace{(1+|x|)^{M+n-N} \cdot |\phi(x)|}^{\leq C_{M+n-N,0}} +$$

$$+ |c_1| \underbrace{(1+|x|)^{M+n-N+1} \cdot |\phi'(x)|}_{\leq C_{M+n-N+1,1}} + \dots + |c_N| \underbrace{(1+|x|)^{M+n} \, |\phi^{(N)}(x)|}_{\leq C_{M+n,N}}$$

$\underline{\underline{\leq \tilde{C}}}$, for every $x$.

$\Longrightarrow$ $\underline{\phi_n \in \mathcal{S}}$.

6) $\phi_n = \phi^{(n)}$ $\dots \rightarrow$ similar.

⑧: Assume $\phi \in \mathcal{S}$.

Want $\hat{\phi} \overset{?}{\in} \mathcal{S}$. ??

$\phi \in C^{\infty}$. $\not\Rightarrow$ $|\hat{\phi}(s)|$ decays faster than any polynomial.

$|x|^{M} \cdot |\phi(x)| \in L^1$ $\Rightarrow$ $\hat{\phi}(s)$ is $C^{\infty}$, differentiable infinitely many times.

$\forall M$.

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$$\phi \longrightarrow \hat{\phi}$$

$$\phi' \longrightarrow 2\pi i s \cdot \hat{\phi}(s).$$

$$x \cdot \phi \longrightarrow \frac{1}{-2\pi i} \frac{d}{ds} \hat{\phi}$$

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Repeat for $x \cdot \phi \dashrightarrow \hat{\phi}' \longrightarrow$ decays faster than any polynomial.

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$$\Rightarrow \hat{\phi} \in \mathcal{S}$$

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# Continuous and Slowly Growing Functions

**Definition**

A continuous function $g: \mathbb{R} \to \mathbb{C}$ is said slowly growing if there is an integer $n \geq 0$ s.t. $\lim_{x \to \pm\infty} \frac{g(x)}{|x|^n} = 0$.

$$CSG = \left\{ g: \mathbb{R} \to \mathbb{C} \mid g \text{ continuous } \& \lim_{x \to \pm\infty} \frac{g(x)}{|x|^n} = 0, \text{ for some } n \geq 0 \right\}$$

**Examples.**

① $g(x) = x + 3x^{22}$

→ is a CSG function:

i) continuous ✓

ii) $\lim_{x \to \pm\infty} \frac{g(x)}{x^{24}} = 0$

$\boxed{\text{Any polynomial is a CSG}}$

② If $g \in \mathcal{S}$ ⟶ $g$ is also a CSG.

$\boxed{\text{Any test function is CSG.}}$

③ $g(x) = \sin(x)$, $g(x) = \cos(x)$ ⟶ are CSG.

④ $g(x) = \sin(e^x)$, $g(x) = \cos(e^x)$ ---> are CSG.

⑤ $\boxed{g(x) = e^x \text{ ---> NOT a CSG}}$. $\boxed{g(x) = e^{a \cdot x} \text{ is NOT CSG}}$ $a \neq 0$

6) $g(x) = e^{-|x|}$ is a CSG.

Proposition:

1) If $g_1, g_2$ are CSG, $a_1, a_2 \in \mathbb{C}$, then $a_1 g_1 + a_2 g_2$ is also CSG

2) If $g_1, g_2$ are CSG then $g_1 \cdot g_2$ is CSG.

[ CSG is an algebra with identity, identity is the constant function 1 ].

Definition. To a CSG function $g : \mathbb{R} \to \mathbb{C}$ we associate its fundamental functional

$$g : \mathcal{S} \longrightarrow \mathbb{C}, \quad g\{\phi\} = \int_{-\infty}^{\infty} g(x) \cdot \phi(x)\, dx$$

Notation: In some books, the fundamental functional is denoted $T_g : \mathcal{S} \to \mathbb{C}$, $T_g(\phi) = \int_{-\infty}^{\infty} g(x)\, \phi(x)\, dx$.

Claim: For any $\phi \in \mathcal{S} \longrightarrow \int_{-\infty}^{\infty} g(x)\, \phi(x)\, dx$ is well-defined and finite.

$g$ is a CSG $\dashrightarrow$ $\exists\, n_0$ integer s.t. $\lim\limits_{x \to \pm\infty} \frac{g(x)}{|x|^{n_0}} = 0$.

On the other hand: $(1 + |x|)^{n_0+2} \cdot |\phi(x)| \leq C_{n_0+2, 0}$, $\forall x$.

Then:
$$\int_{-\infty}^{\infty} |g(x)\phi(x)|dx = \int_{-\infty}^{\infty} |g(x)\cdot\phi(x)|\, dx =$$

$$= \int_{-\infty}^{\infty} \underbrace{\left|\frac{g(x)}{(1+|x|)^{n_0}}\right|} \cdot \underbrace{\left|(1+|x|)^{n_0+2}\cdot|\phi(x)|\right|}_{\leq C_{n_0+2,0}} \cdot \frac{1}{(1+|x|)^2}\, dx. \leq$$

bounded: $\frac{g(x)}{(1+|x|)^{n_0}} \leq D_1$, for every $x$.

$$\leq D_1 \cdot C_{n_0+2,0} \cdot \int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^2} = 2\, D_1\, C_{n_0+2,0} \underbrace{\int_0^{\infty} \frac{dx}{(1+x)^2}}_{-\frac{1}{1+x}\Big|_0^{\infty}} =$$

$$= 2\, D_1 \cdot C_{n_0+2,0}. < \infty.$$

$\Rightarrow \int_{-\infty}^{\infty} g(x)\,\phi(x)\,dx$ is (absolutely convergent) finite.

<u>Claim 2</u>: If $g \in CSG$, $\phi_1, \phi_2 \in \mathcal{S}$, $a_1, a_2 \in \mathbb{C}$ $\Big| \rightarrow g\{a_1\phi_1 + a_2\phi_2\} = a_1\, g\{\phi_1\} + a_2\, g\{\phi_2\}$.

$\Rightarrow g\{\cdot\}$ is a linear function on $\mathcal{S}$.

Why to introduce functionals?

Assume $g$ is CSG such that: 1) $g$ is $C^1$

additionally: 2) $g'$ is also a CSG.

Then:

$$g'\{\phi\} = \int_{-\infty}^{\infty} g'(x)\,\phi(x)\,dx \underset{\substack{\uparrow \\ \text{integration} \\ \text{by parts.}}}{=} \underbrace{g(x)\cdot\phi(x)\Big|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} g(x)\cdot\phi'(x)\,dx$$

$$\lim_{x\to\pm\infty} (g(x)\cdot\phi(x)) = \lim_{x\to\pm\infty}\Bigg(\underbrace{\frac{g(x)}{|x|^{n_0}}}_{\substack{\downarrow\ x\to\pm\infty \\ 0}} \cdot \underbrace{|x|^{n_0}\,|\phi(x)|}_{\text{bounded.}}\Bigg) = 0.$$

Thus:

$$\underline{g'\{\phi\}} = -\int_{-\infty}^{\infty} g(x)\cdot\phi'(x)\,dx = \underline{-g\{\phi'\}}.$$

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<u>Definition</u>: 1) Assume $g$ is a CSG.

Then the <u>derivative</u> of its associated functional is:

defined by:

$$g'\{\phi\} = -g\{\phi'\}.$$

2) $g' : \mathcal{S} \to \mathbb{C}$, $g'\{\phi\} = -g\{\phi'\}$

is also called the derivative of $g$ in the sense of distributions

Formally, we denote by $g'$ a "fictious" function

s.t. $$g'\{\phi\} := \int_{-\infty}^{\infty} g'(x)\cdot\phi(x)\,dx = -g\{\phi'\}.$$

3). The $n^{th}$ derivative of $g$:

$$g^{(n)}\{\phi\} = (-1)^n \cdot g\{\phi^{(n)}\}.$$

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__Definition__. A linear functional $F: \mathcal{S} \longrightarrow \mathbb{C}$ is called a generalized function (or, distribution) if there exists a CSG $g: \mathbb{R} \to \mathbb{C}$ and an integer $N \geq 0$ such that,

$$F\{\phi\} = (-1)^N \cdot g\{\phi^{(N)}\} = (-1)^N \cdot \int_{-\infty}^{\infty} g(x)\cdot\phi^{(N)}(x)\,dx.$$

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__Notation__: $F = g^{(N)}$ in the sense of distributions.

__Definition__.

$$F'\{\phi\} = -F\{\phi'\}.$$ ← derivative of the distribution $F$

$$F''\{\phi\} = +F\{\phi''\}.$$ ← $2^{nd}$ derivative of $F$

...

$$F^{(M)}\{\phi\} = (-1)^M F\{\phi^{(M)}\}.$$ ← $M^{th}$ derivative of $F$.