L17

Some Elementary Generalized Functions

Example.

If $g(x) = \sin(x)$

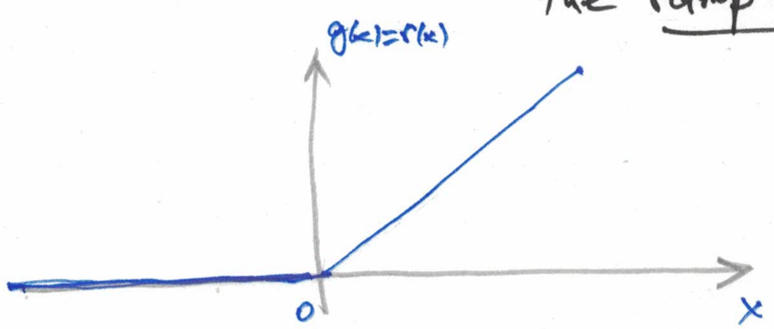
$\phi(x) = e^{-x^2}$

Then $$g\{\phi\} = \int_{-\infty}^{\infty} \sin(x)\, e^{-x^2}\, dx = \dots = 0.$$

Construction of Dirac's Delta Generalized Function.

Consider $g: \mathbb{R} \to \mathbb{R}$, $$g(x) = \begin{cases} x, & x \geq 0. \\ 0, & x < 0. \end{cases}$$

the ramp function, denoted by r.



Note: 1. g is continuous on $\mathbb{R}$.

2. $\lim_{x \to \pm\infty} \frac{g(x)}{x^2} = 0.$

$\to$ g is a CSG function

Take $\phi \in \mathcal{S}$

$$g\{\phi\} := \int_{-\infty}^{\infty} g(x)\,\phi(x)\,dx = \int_{0}^{\infty} x \cdot \phi(x)\,dx$$

Let $h = g'$, the derivative in the sense of distributions.

$h\{\phi\} = ?$

$$h\{\phi\} = g'\{\phi\} \overset{\text{by definition}}{=} -g\{\phi'\} = -\int_0^\infty x\cdot\phi'(x)\,dx \overset{\substack{\text{integration by}\\ \text{parts}}}{=}$$

$$= -\underbrace{\cancel{x\cdot\phi(x)}\Big|_0^\infty}_{0} + \int_0^\infty \phi(x)\,dx = \int_0^\infty \phi(x)\,dx = \int_{-\infty}^{\infty} h(x)\cdot\phi(x)\,dx.$$

$$\lim_{x\to\infty} x\,\phi(x) = 0$$

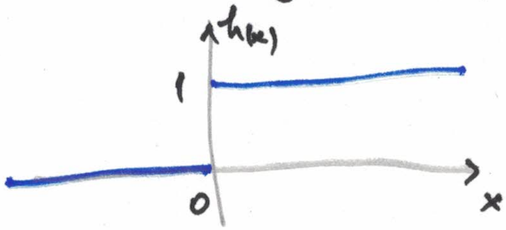
$$x\cdot\phi(x)\Big|_{x=0} = 0.$$

where: $h : \mathbb{R} \to \mathbb{R}$,

$$h(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0. \end{cases}$$

This function is called the unit step function, or the Heaviside (step) function

$g' = h$, in the sense of distributions!



Note: h is NOT a CSG function. because it is discontinuous at $x=0$.

$h : \mathbb{R} \to \mathbb{R}$ is an ordinary function.

However we can associate a fundamental functional

via: $\phi \longmapsto h\{\phi\} = \int_{-\infty}^{\infty} h(x)\,\phi(x)\,dx = \int_0^\infty \phi(x)\,dx$.

Compute $g'' = h'$:

Take $\phi \in \mathcal{J}$,

$$h'\{\phi\} = -h\{\phi'\} = -\int_0^\infty \phi'(x)\,dx = -\left(\phi(x)\Big|_0^\infty\right) = -\underbrace{\lim_{x\to\infty}\phi(x)}_{0} + \phi(0)$$

$$\Rightarrow h'\{\phi\} = \phi(0)$$

<u>Definition</u> $\delta = h' = g''$ is the <u>Dirac's delta distribution</u>.

<u>"Formally"</u>: $\delta\{\phi\} = \phi(0)$.

$$\underline{\delta\{\phi\}} = \underbrace{\int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx}_{\text{Formal integral.}} = \phi(0).$$

[<u>NOTE</u>: There is no <u>ordinary function</u> $\delta : \mathbb{R} \to \mathbb{R}$ such that.
for every $\phi \in \mathcal{S}$, $\int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx = \phi(0)$.]

<u>Derivatives, Translates and Scaling of δ.</u>

δ' : $\delta'\{\phi\} = -\delta\{\phi'\} = -\phi'(0) \xleftrightarrow{\text{Formal}} \int_{-\infty}^{\infty} \delta'(x)\,\phi(x)\,dx$

δ'' : $\delta''\{\phi\} = \delta\{\phi''\} = \phi''(0) \longleftrightarrow \int_{-\infty}^{\infty} \delta''(x)\,\phi(x)\,dx$

$$\boxed{\delta^{(n)}\{\phi\} = (-1)^n \,\phi^{(n)}(0).} \longleftrightarrow \underbrace{\int_{-\infty}^{\infty} \delta^{(n)}(x)\,\phi(x)\,dx}_{\text{Formal integral}}$$

<u>Example</u>:

$$\int_{-\infty}^{\infty} \delta''(x)\, e^{-x^2}\,dx = \frac{d^2}{dx^2}\left(e^{-x^2}\right)\Big|_{x=0} = \frac{d}{dx}\left(-2x e^{-x^2}\right)\Big|_{x=0} =$$

$$= \left(-2e^{-x^2} + 4x^2 e^{-x^2}\right)\Big|_{x=0} = -2.$$

Translation

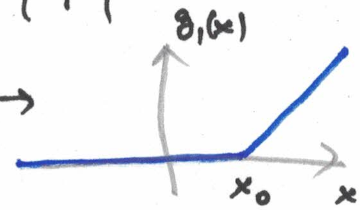
What is $\delta(x-x_0) = ?$, for some $x_0 \in \mathbb{R}$.

$$\int_{-\infty}^{\infty} \delta(x-x_0)\,\phi(x)\,dx = \int_{-\infty}^{\infty} \delta(y)\cdot\phi(y+x_0)\,dy = \phi(0+x_0) = \phi(x_0).$$

$y = x - x_0$.

It is a distribution because: it is equal to $\cancel{\delta}\ g_1''\{\phi\}$

where $g_1(x) = r(x-x_0) \longrightarrow$



General rule:

If f denotes a distribution : $\phi \longmapsto f\{\phi\}$.

and $x_0 \in \mathbb{R}$.

Then $g(x) = f(x-x_0)$ defines a new distribution

given by: $g\{\phi\} = f\{\psi\}$, where $\psi(x) = \phi(x+x_0)$.

Scaling: Fix $a \neq 0$. Want $\delta(a\cdot x) = ?$

$$\int_{-\infty}^{\infty} \delta(a\cdot x)\,\phi(x)\,dx = \int_{-\infty}^{\infty} \delta(y)\cdot\phi\left(\frac{y}{a}\right)\frac{1}{|a|}\,dy = \frac{\phi(0)}{|a|} =$$

$y = a\cdot x$

$dy = |a|\,dx$

$$= \int_{-\infty}^{\infty} \frac{1}{|a|}\delta(x)\,\phi(x)\,dx$$

We obtained:
$$\int_{-\infty}^{\infty} \delta(a\cdot x)\, \phi(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{|a|}\, \delta(x)\, \phi(x)\, dx .$$

$$\boxed{\delta(a.x) = \frac{1}{|a|}\, \delta(x).}$$

<u>General Rule</u>: If f denotes a distribution : $\phi \longmapsto f\{\phi\}$

and $a \in \mathbb{R}$, $a \neq 0$

Then $g(x) = f(a\cdot x)$ defines a new distribution

given by $g\{\phi\} = f\{\psi\}$, where $\psi(x) = \frac{1}{|a|}\phi\left(\frac{x}{a}\right)$.

<u>Example</u>:

$$\int_{-\infty}^{\infty} \delta(5x-2)\, \phi(x)\, dx = \int_{-\infty}^{\infty} \delta(y)\, \phi\left(\frac{y+2}{5}\right) \frac{1}{5}\, dy = \frac{1}{5}\, \phi\left(\frac{2}{5}\right).$$

$y = 5x-2.$

$$= \frac{1}{5} \int_{-\infty}^{\infty} \delta\left(x - \frac{2}{5}\right) \phi(x)\, dx$$

We obtained: $\delta(5x-2) = \frac{1}{5}\, \delta\left(x - \frac{2}{5}\right)$

<u>Example</u>.

$$\int_{-\infty}^{\infty} \delta'(2x+1)\, \phi(x)\, dx = \int_{-\infty}^{\infty} \delta'(y)\, \phi\left(\frac{y-1}{2}\right) \frac{1}{2}\, dy = -\int_{-\infty}^{\infty} \delta(y) \cdot \frac{d}{dy}\left(\phi\left(\frac{y-1}{2}\right)\right) \frac{1}{2}\, dy$$

$y = 2x+1$

$dy = 2\, dx$

$$= -\frac{1}{2} \int_{-\infty}^{\infty} \delta(y) \cdot \phi'\left(\frac{y-1}{2}\right) \cdot \frac{1}{2}\, dy = -\frac{1}{4}\, \phi'\left(-\frac{1}{2}\right).$$

Thus: $$\boxed{\int_{-\infty}^{\infty} \delta'(2x+1)\,\phi(x)\,dx = -\frac{1}{4}\phi'\left(-\frac{1}{2}\right)}$$

$$= \frac{1}{4}\int_{-\infty}^{\infty}\delta'\left(x+\frac{1}{2}\right)\phi(x)\,dx$$

and: $$\underline{\delta'(2x+1) = \frac{1}{4}\,\delta'\left(x+\frac{1}{2}\right).}$$

The power function

As an ordinary function: $p_n(x) = x^n$, $n = 0, 1, 2, \ldots$

As distribution:

$$p_n\{\phi\} = \int_{-\infty}^{\infty} x^n \cdot \phi(x)\,dx.$$

Derivative in the sense of distributions:

integrating by parts.

$$p_n'\{\phi\} = -p_n\{\phi'\} = -\int_{-\infty}^{\infty} x^n \cdot \phi'(x)\,dx =$$

$$= \underbrace{-x^n \cdot \phi(x)\Big|_{-\infty}^{\infty}}_{\lim\limits_{x\to\pm\infty} x^n\cdot\phi(x) = 0} + \int_{-\infty}^{\infty} n \cdot x^{n-1} \cdot \phi(x)\,dx = \int_{-\infty}^{\infty} n x^{n-1}\,\phi(x)\,dx.$$

We obtained:

$$\underline{p_n' = n \cdot p_{n-1}}.$$

(Compatible with $(x^n)' = n \cdot x^{n-1}$).

What about $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$, ... ?

Construction of P_{-1}

, P_{-1} : distribution associated to $\frac{1}{x}$

Let $g: \mathbb{R} \to \mathbb{R}$, $g(x) = \begin{cases} x \cdot \log(|x|) - x, & x \neq 0 \\ 0, & x = 0. \end{cases}$

($\log = \ln$, is the natural logarithm).

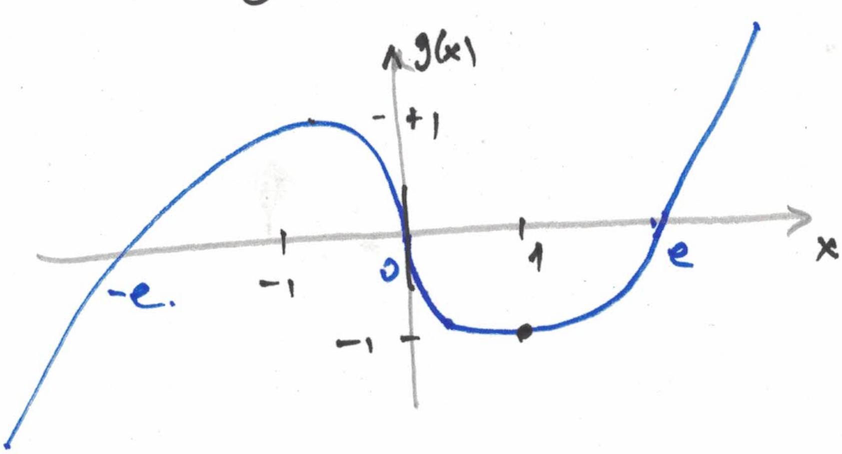
1) $$\lim_{x \to 0} g(x) = \lim_{x \to 0} (x \log(|x|) - x) = \lim_{x \downarrow 0} \frac{\log(x)}{\frac{1}{x}} \overset{\text{l'Hospital}}{=} \lim_{x \downarrow 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} =$$

$$= \lim_{x \downarrow 0} (-x) = 0 = g(0).$$

$\Rightarrow g$ continuous.

2) $$\lim_{x \to \pm\infty} \frac{g(x)}{x^2} = \lim_{x \to \pm\infty} \left(\frac{\log(|x|)}{x} - \frac{1}{x} \right) \overset{\text{l'Hospital}}{=} \lim_{x \to \pm\infty} \frac{\frac{1}{x}}{1} = 0.$$

$\Rightarrow g$ is a CSG function.



For $x > 0$, $g(x) = x \log(x) - x$

$g'(x) = \log(x)$.

$g'(x) = 0 \Leftrightarrow x = 1$

What is g' in the sense of distributions?

$$g'\{\phi\} = -g\{\phi'\} = -\int_{-\infty}^{\infty}\left(x\log(|x|)-x\right)\phi'(x)\,dx =$$

$$= -\int_{-\infty}^{0}\left(x\log(-x)-x\right)\phi'(x)\,dx - \int_{0}^{\infty}\left(x\log(x)-x\right)\phi'(x)\,dx =$$

$$= -\underbrace{(x\log(-x)-x)\phi(x)\Big|_{-\infty}^{0}}_{0} + \int_{-\infty}^{0}\underbrace{\frac{d}{dx}\left(x\log(-x)-x\right)}_{1\cdot\log(-x)+x\cdot\frac{1}{-x}(-1)-1=\log(-x)}\phi(x)\,dx -$$

$$-\underbrace{(x\log(x)-x)\phi(x)\Big|_{0}^{\infty}}_{0} + \int_{0}^{\infty}\frac{d}{dx}\left(x\log(x)-x\right)\phi(x)\,dx =$$

$$= \int_{-\infty}^{0}\log(-x)\,\phi(x)\,dx + \int_{0}^{\infty}\log(x)\,\phi(x)\,dx =$$

$$= \int_{-\infty}^{\infty}\log(|x|)\,\phi(x)\,dx.$$

We obtained: $\underline{g' = \log(|x|)}$. in the sense of distribution:

$$g'\{\phi\} = \int_{-\infty}^{\infty}\log(|x|)\cdot\phi(x)\,dx$$

Note: $\log(|x|)$ is NOT a CSG. and the integral has an (apparent) singularity at 0