Operations with Distributions

Operations:

1) Assume f and g are two distributions, and $a, b \in \mathbb{C}$
then $a \cdot f + b \cdot g$ (linear combination) is the distribution:

$$(a \cdot f + b \cdot g)\{\phi\} = a \cdot f\{\phi\} + b \cdot g\{\phi\}, \quad \forall \phi \in \mathcal{J}$$

2) Assume. $x_0 \in \mathbb{R}$. Given f a distribution,
then $g(x) = f(x - x_0)$ is the distribution.

$$\forall \phi \in \mathcal{J}, \quad g\{\phi\} = f\{\psi\}, \qquad \psi(x) = \phi(x + x_0),$$

$$\left(\text{It mimick: } \int_{-\infty}^{\infty} f(x - x_0)\, \phi(x)\, dx = \int_{-\infty}^{\infty} f(y)\, \phi(y + x_0)\, dy \right).$$

3). Assume f is a distribution and $a \in \mathbb{R}, a \neq 0$.
then $g(x) = f(a \cdot x)$ defines the distribution:

$$\forall \phi \in \mathcal{J}, \quad g\{\phi\} = f\{\psi\}, \qquad \psi(x) = \frac{1}{|a|}\, \phi\left(\frac{x}{a}\right).$$

$$\left(\text{It mimicks: } \int_{-\infty}^{\infty} f(a \cdot x)\, \phi(x)\, dx = \int_{-\infty}^{\infty} f(y)\, \phi\left(\frac{y}{a}\right) \frac{1}{|a|}\, dy \right).$$

4) Assume f is a distribution and $\omega_0 \in \mathbb{R}$, then

$$g(x) = e^{2\pi i \omega_0 x} \cdot f(x) \quad \text{defines the distribution,}$$

$$\forall \phi \in \mathcal{S}, \quad g\{\phi\} = f\{\psi\}, \quad \psi(x) = e^{2\pi j \omega_0 x} \cdot \phi(x).$$

(It mimicks: $\int_{-\infty}^{\infty} e^{2\pi i \omega_0 x} f(x) \phi(x) dx = \int_{-\infty}^{\infty} f(x) \left(e^{2\pi i \omega_0 x} \cdot \phi(x)\right) dx$).

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5). Assume f is a distribution and $P(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n$ a polynomial, then

$$g(x) = P(x) \cdot f(x) \quad \text{defines the distribution}$$

$$\forall \phi \in \mathcal{S}, \quad g\{\phi\} = f\{\psi\}, \quad \text{where } \psi(x) = P(x) \cdot \phi(x)$$

(It mimicks: $\int_{-\infty}^{\infty} P(x) f(x) \phi(x) dx = \int_{-\infty}^{\infty} f(x) \left(P(x) \phi(x)\right) dx$)

Why: $g(x) = x \cdot f(x)$ is a distribution?

Assume $f = u'$, is the 1st order derivative in the sense of distributions, where u is a CSG function.

Want: Need to find a V_1, V_2 CSG* functions such that $g = V_1' + V_2$.

$$g(x) = x \cdot f(x) = x \cdot u'(x) = \frac{d}{dx}\underbrace{(x \cdot u(x))}_{v_1} - \underbrace{u(x)}_{v_2} = v_1' + v_2 .$$

Note: $v_1(x) = x \cdot u(x)$, $v_2(x) = -u(x)$ are CSG functions.

Why:

1) If u continuous $\Rightarrow$ $x \cdot u$, $-u$ are continuous.

2) If $N_0 \geq 0$ integer s.t. $\lim_{x \to \pm\infty} \frac{u(x)}{|x|^{N_0}} = 0$

then. $\lim_{x \to \pm\infty} \frac{v_1(x)}{|x|^{N_0+1}} = 0 = \lim_{x \to \pm\infty} \frac{v_2(x)}{|x|^{N_0+1}}$

Alternatively: Let $W(x) = \int_0^x v_2(y)\,dy$. (an antiderivative)

Since i) v_2 is cont. $\longrightarrow$ W is continuous.

ii). If. $\lim_{x \to \pm\infty} \frac{v_2(x)}{|x|^{N_0}} = 0 \Rightarrow \lim_{x \to \pm\infty} \frac{W(x)}{|x|^{N_0+1}} = 0.$

$\longrightarrow v_2 = W' \Rightarrow g(x) = v_1' + v_2 = \underbrace{(v_1 + W)}_{CSG}'$

<u>One interesting example:</u> The "comb" distribution.

$$Ш(x) = \sum_{n=-\infty}^{\infty} \delta(x-n).$$

Its action:

$$Ш\{\phi\} = \sum_{n=-\infty}^{\infty} \phi(n). \quad , \forall \phi \in \mathcal{S}.$$

It mimicks: $\displaystyle\int_{-\infty}^{\infty} \Big(\sum_{n=-\infty}^{\infty} \delta(x-n)\Big) \cdot \phi(x)\, dx = \sum_{n=-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} \delta(x-n)\phi(x)\, dx = \sum_{n=-\infty}^{\infty} \phi(n)\Big).$

[How to show that. $\quad \phi \mapsto Ш\{\phi\} = \displaystyle\sum_{n=-\infty}^{\infty} \phi(n)$

defines a distribution ?

<u>Need</u> to find: 1) a CSG., f

2) an integer $N \geq 0$

such that $\quad f^{(N)}\{\phi\} = \displaystyle\sum_{n=-\infty}^{\infty} \phi(n). \quad , \forall \phi \in \mathcal{S}.$

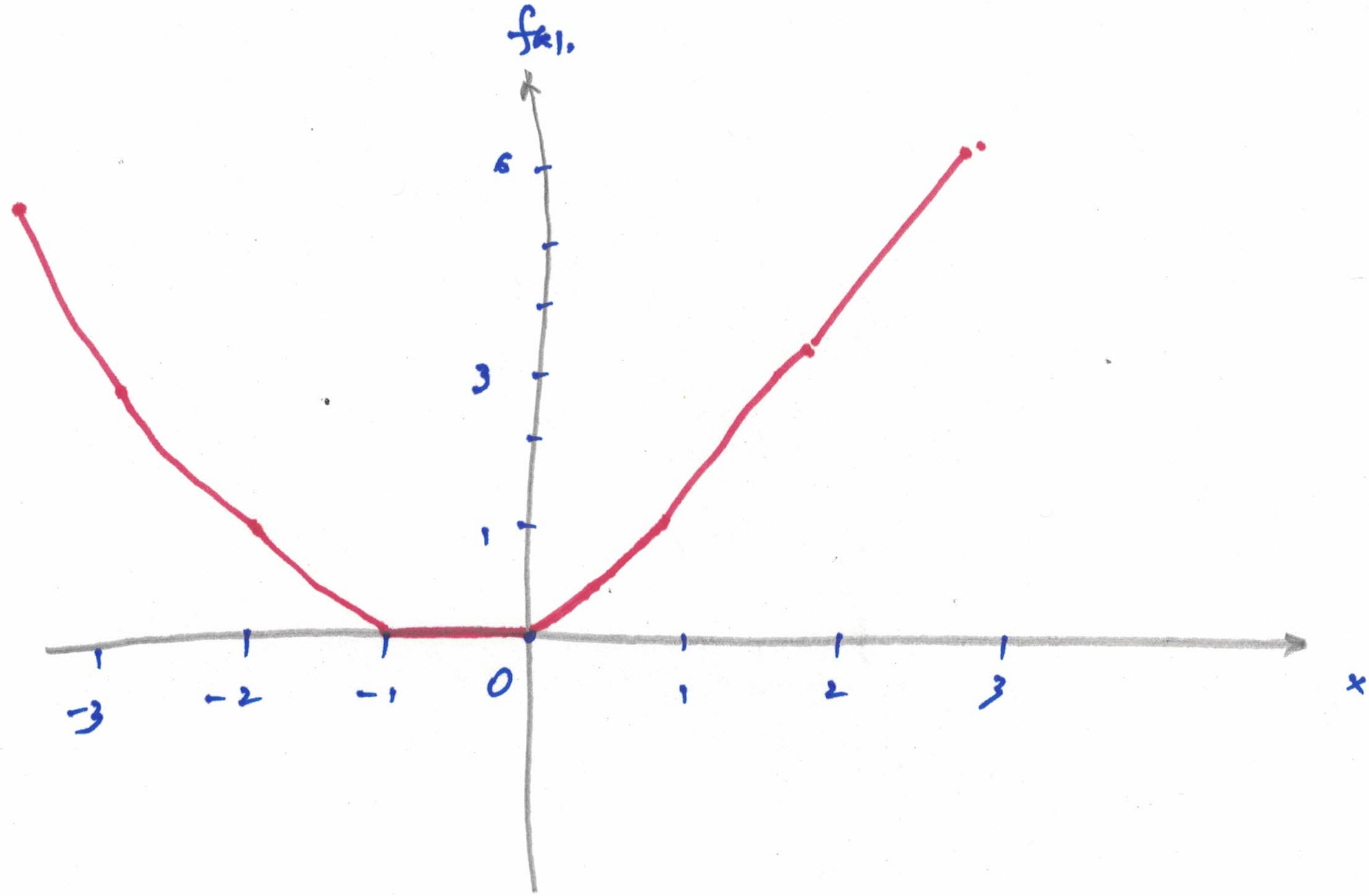
Recall: $\quad \delta = g''$, where $g(x) = \begin{cases} x, & x \geq 0. \\ 0, & x < 0. \end{cases} \rightarrow$ CSG.

$$\delta(x-n) = (g(x-n))'' = (g(x-n) + ax + b)''$$

The construction: Set $N=2$.

Need to produce $f:\mathbb{R}\to\mathbb{R}$, a CSG s.t. f is piecewise linear

" and. $f'' = \sum_{n=-\infty}^{\infty} \delta(\cdot - n)$ ".



$$f(x) = \begin{cases} (n+1)\cdot(x-n) + 1+2+\dots+n, & n \le x < n+1 \\ \vdots & \\ 3(x-2)+1+2, & 2 \le x < 3 \\ 2(x-1)+1, & 1 \le x < 2. \\ x, & 0 \le x < 1. \\ 0, & -1 \le x < 0. \\ -(x+2)+1, & -2 \le x < -1 \\ \vdots & \end{cases}$$

for $n \le x < n+1$

$\Rightarrow$ $(n+1)\cdot(x-n)+1+2+\dots+n$

$= (n+1)x - n(n+1) + \frac{n(n+1)}{2} =$

$= (n+1)x - \frac{n(n+1)}{2}$

$\Rightarrow$ $f(x) = (n+1)x - \frac{n(n+1)}{2}$, for $n \le x < n+1$

Why is this f a CSG:

1) Continuity follows: 1) piecewise linear.

2).

Fix n: $\underbrace{\lim_{x \uparrow n} f(x)}$ $=$ $\underbrace{\lim_{x \downarrow n} f(x)}$

$$(n+1)\cdot n - \frac{n(n+1)}{2} = \frac{n(n+1)}{2}$$

$$\left.(n-1+1)x - \frac{(n-1)\cdot(n-1+1)}{2}\right|_{x=n} = n^2 - \frac{n(n-1)}{2} = \frac{n(2n-n+1)}{2} = \frac{n\cdot(n+1)}{2}$$

2).

$$n \leq x < n+1 \dashrightarrow f(x) = (n+1)x - \frac{n(n+1)}{2} \leq$$

$$\leq (x+1)x - \frac{(x-1)x}{2} =$$

$$= x^2 + x - \frac{x^2-x}{2} = \frac{x^2}{2} + \frac{3x}{2}$$

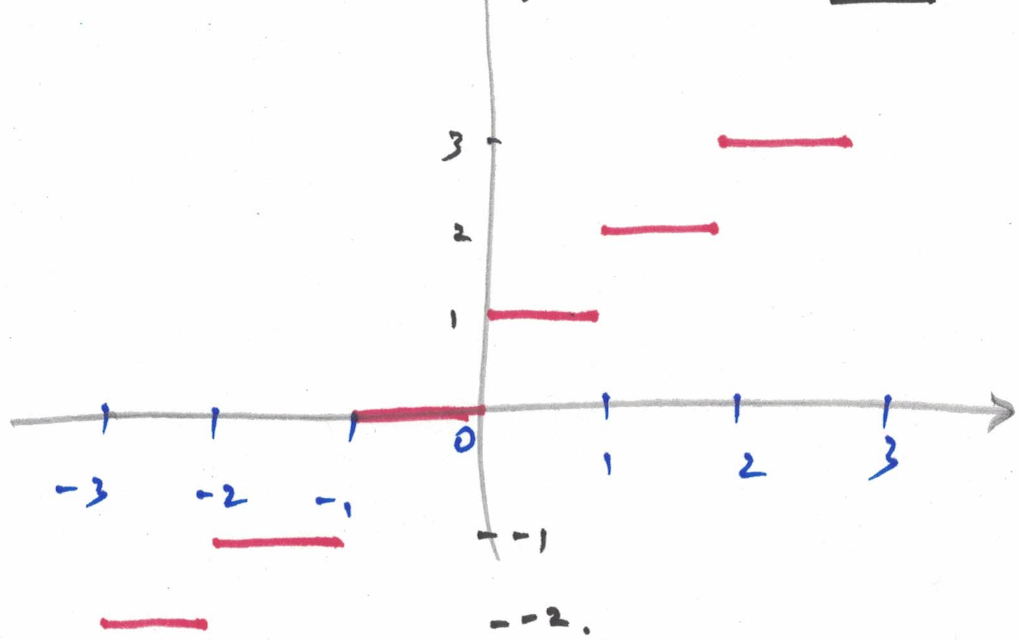
Thus for every $x \in \mathbb{R}$,

$$0 \leq f(x) \leq \frac{x^2}{2} + \frac{3x}{2}$$

Then: $$\lim_{x \to \pm\infty} \frac{f(x)}{|x|^3} = 0.$$

Why $f'' = \sum \delta(x-n) \dashrightarrow f'$:

f' : at all x <u>except</u> integers.



$f'(x) = n+1$, for $n < x < n+1$. $\rightarrow$ piecewise constant.

$$f''\{\phi\} = -f'\{\phi'\} = -\sum_{n=-\infty}^{\infty}\int_{n}^{n+1} f'(x)\,\phi'(x)\,dx =$$

$$= -\sum_{n=-\infty}^{\infty}(n+1)\cdot\int_{n}^{n+1}\phi'(x)dx = -\sum_{n=-\infty}^{\infty}(n+1)\cdot\big(\phi(n+1)-\phi(n)\big) =$$

$$= -\Big[\overbrace{\dots + -1(\phi(-1)-\phi(-2))}^{-(\phi(-1)-\phi(-2))} + \phi(1)-\phi(0) + 2\cdot\big(\phi(2)-\phi(1)\big) + 3\cdot\big(\phi(3)-\phi(2)\big) + \dots\Big]$$

$$= -\big[\dots -\phi(-2) \; -\phi(-1) \; -\phi(0) - \phi(1). - \phi(2) \; - \dots \big]$$

$$= \dots \phi(-2) + \phi(-1) + \phi(0) + \phi(1) + \phi(2) + \dots$$

Multiplication of Distributions.

Assume. f is a distribution and. $\varphi : R \to \mathbb{C}$ is an ordinary functions such that: $\forall \phi \in \mathcal{S} \longmapsto \varphi \cdot \phi \in \mathcal{S}$.

Then $g = \varphi \cdot f$ defines the distribution:

$$g\{\phi\} = f\{\varphi \cdot \phi\}.$$

(It mimicks: $\int_{-\infty}^{\infty} \varphi(x) \cdot f(x) \cdot \phi(x) dx = \int_{-\infty}^{\infty} f(x) \cdot (\varphi(x) \cdot \phi(x)) dx$).

Convolution of Distributions

Definition Assume f and g are two distributions.
Assume also the following:

1). For any test function $\phi \in \mathcal{S}$, and $x \in R$

let $\psi_x(y) = \phi(x+y)$. Note $\psi_x \in \mathcal{S}$.

and define. $\underline{\underline{\alpha}} : R \to \mathbb{C}$, $\alpha(x) = g\{\psi_x\}$.

$\boxed{\text{Assumption}}$: $\alpha \in \mathcal{S}$

2). Definition: $f * g$ is the distribution.

$$\forall \phi \in \mathcal{S} \quad f * g\{\phi\} = f\{\alpha\}.$$

It mimicks :

$$f * g\,(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy = \int_{-\infty}^{\infty} f(y) \cdot g(x-y)\, dy .$$

$$(f*g)\{\phi\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy \;\; \phi(x)\, dx . =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy \;\; \phi(x)\, dx =$$

$$= \int_{-\infty}^{\infty} \Big(\underbrace{\int_{-\infty}^{\infty} g(x-y)\, \phi(x)\, dx}_{\alpha(y)} \Big) f(y)\, dy = \int_{-\infty}^{\infty} f(y)\, \alpha(y)\, dy = f\{\alpha\}.$$

$$\alpha(y) = \int_{-\infty}^{\infty} g(x-y)\, \phi(x)\, dx$$

$$\alpha(x) = \int_{-\infty}^{\infty} g(y-x)\, \phi(y)\, dy = \int_{-\infty}^{\infty} g(y)\, \underbrace{\phi(y+x)}_{\psi_x}\, dy = g\{\psi_x\}.$$

where $\psi_x(y) = \phi(y+x)$.

$y \mapsto t = y - x$

$$\int_{-\infty}^{\infty} g(t)\, \phi(t+x)\, dt = \int_{-\infty}^{\infty} g(y)\, \phi(y+x)\, dy$$