MATH 464: Transform Methods.

Notations & Review of Calculus, Complex Numbers, Linear Algebra.

Numbers:

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ set of natural numbers.

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ set of integers

$\mathbb{Q} = \{\frac{p}{q},\ p \in \mathbb{Z},\ q \in \mathbb{N},\ q \neq 0\}$ set of rational numbers

$\hookrightarrow \in$ "belongs to".

"is in"

$\mathbb{R}$ = set of real numbers $= (-\infty, \infty)$

$\mathbb{C} = \{x + iy\ ;\ x, y \in \mathbb{R}\}$ set of complex numbers

$i = \sqrt{-1}$. $\quad (i^2 = -1)$.

Addition $+$, Multiplication $\cdot$ $\longrightarrow$ internal operations

In $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$: opposite element : $x \longmapsto -x$.

$x + (-x) = 0$

In $\mathbb{Q}, \mathbb{R}, \mathbb{C}$: inverse element : $x \neq 0 \longmapsto \frac{1}{x}$

$x \cdot \frac{1}{x} = 1$.

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

countable $\leftarrow | \rightarrow$ uncountable.

$(\mathbb{Z}, +)$ --→ abelian group

commutative group.

$+ : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$. $(a,b) \longmapsto a+b$. : internal operation.

Properties:

1) associativity: $(a+b)+c = a+(b+c)$

2). neutral element: $a+0 = 0+a = a$

3). opposite element: $a+(-a) = (-a)+a = 0$

4). commutativity: $a+b = b+a$

$(\mathbb{R}, +, \cdot)$. --→ field.

1) $(\mathbb{R}, +)$ is a commutative group.

2) $(\mathbb{R} \setminus \{0\}, \cdot)$ is a commutative group.

3). distributivity $a \cdot (b+c) = a \cdot b + a \cdot c$.

4) $1 \neq 0$.

$(\mathbb{Q}, +, \cdot)$ --→ field.

Additionally: "$\leq$" less than or equal

$(\mathbb{R}, +, \cdot, \leq)$, $(\mathbb{Q}, +, \cdot, \leq)$ totally ordered fields.

Unlike $\mathbb{Q}$, $\mathbb{R}$ is a complete space.
→ $\boxed{\text{topologically complete}}$

Definition A sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}$ is said convergent in $\mathbb{R}$.

if there exists $z \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = z$.

For every $\varepsilon > 0$ there exists a positive integer N that may depend on ε such that for every integer $n \geq N$, $|x_n - z| < \varepsilon$.

$[\forall \varepsilon > 0 \ \exists N_\varepsilon > 0 \ \forall n \geq N_\varepsilon, \ |x_n - z| < \varepsilon.]$ Formal writing.

$\forall$: For every (For any). $\exists$: There is (There exists)

Definition A sequence $(x_n)_{n\geq 1}$ is said Cauchy

if,

For every $\varepsilon > 0$ there exists a positive integer N that may depend on ε

such that for any integers $n, m \geq N$, $|x_n - x_m| < \varepsilon$.

$[\forall \varepsilon > 0 \ \exists N_\varepsilon > 0 \ \forall n, m \geq N_\varepsilon, \ |x_n - x_m| < \varepsilon.]$

Property. Any convergent sequence is Cauchy.

Definition A set S of numbers (integers, rationals, real, or complex) is said **topologically complete** (or just **complete**) if every Cauchy sequence in S is convergent in S.

Specifically:

[If $(x_n)_{n\geq 1}$, $x_n \in S$, $\forall n$ and $(x_n)_{n\geq 1}$ is Cauchy then there exists $z \in S$ such that $\lim\limits_{n\to\infty} x_n = z$.].

Remark: $\mathbb{Q}$ is NOT complete.

$\mathbb{R}$ is complete.

Algebra of complex numbers.

$z \in \mathbb{C} \longrightarrow z = x + iy$, for some (unique) $x, y \in \mathbb{R}$

x: real part of z

y: imaginary part of z.

$$(x+iy) + (u+iv) = x+u + i(y+v)$$

$$(x+iy)\cdot(u+iv) = x\cdot u - y\cdot v + i(x\cdot v + y\cdot u)$$

$(\mathbb{C}, +, \cdot) \longrightarrow$ field, is a complete field.

$z \in \mathbb{C} \longrightarrow$ complex conjugate $\bar{z} = x - iy$

$z = x + iy$

$$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{x^2 + y^2}$$, (absolute value.) (magnitude of z.).

$z = x + iy$

A sequence. $(z_n)_{n \geq 1}$ in $\mathbb{C}$ is convergent to w in $\mathbb{C}$

if:

$$\forall \varepsilon > 0 \ \exists N_\varepsilon > 0 \ \forall n > N_\varepsilon, \ \underbrace{|z_n - w|}_{\sqrt{(\mathrm{Real}(z_n) - \mathrm{Real}(w))^2 + (\mathrm{Im}(z_n) - \mathrm{Im}(w)}} < \varepsilon.$$

Fact:

$z_n = x_n + i y_n$, $(x_n)_{n \geq 1}$, $(y_n)_{n \geq 1}$ real sequences

$w = u + iv$

$$\lim_{n \to \infty} z_n = w \quad \text{if and only if} \quad \begin{cases} \lim\limits_{n \to \infty} x_n = u \\ \lim\limits_{n \to \infty} y_n = v. \end{cases}$$

$$\frac{1}{x + iy} = \frac{x - iy}{(x+iy)(x-iy)} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}, \quad \left[\begin{matrix} x \neq 0 \\ \text{or} \\ y \neq 0 \end{matrix}\right].$$

$$-(x + iy) = -x - i \cdot y$$

Euler Formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \quad \theta \in \mathbb{R}.$$

$$\left.\begin{aligned} z &= x + iy. \\ \bar{z} &= x - iy. \end{aligned}\right\} \longleftrightarrow \begin{aligned} x &= \mathrm{Re}(z) = \frac{1}{2}(z+\bar{z}) = \frac{z+\bar{z}}{2} \\ y &= \mathrm{Im}(z) = \frac{1}{2i}(z-\bar{z}) = \frac{z-\bar{z}}{2i} \end{aligned}$$

$$\overline{e^{i\theta}} = \cos(\theta) - i\sin(\theta) = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$$

$$\overline{\exp(i\theta)} = \exp(-i\theta)$$

$$\begin{aligned} e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\ e^{-i\theta} &= \cos(\theta) - i\sin(\theta) \end{aligned} \longleftrightarrow \left\{\begin{aligned} \cos(\theta) &= \frac{e^{i\theta}+e^{-i\theta}}{2} \\ \sin(\theta) &= \frac{e^{i\theta}-e^{-i\theta}}{2i} \end{aligned}\right.$$

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$$

$$e^{x+iy} = e^x \cdot e^{iy} = e^x \cdot (\cos(y) + i\sin(y)) =$$

$x, y \in \mathbb{R}$

$$= e^x\cos(y) + i\, e^x \sin(y)$$

$$\overline{e^{x+iy}} = e^x\cos(y) - i\, e^x\sin(y) = e^x\cos(-y) + i\, e^x\sin(-y) = e^{x-iy}$$

$z = x+iy$, $\overline{e^z} = e^{\bar{z}}$

$$\overline{e^{z}} = e^{\bar{z}}$$

Facts:

1. If $z, w \in \mathbb{C}$: $\quad |z+w| \le |z| + |w|$
2. If $z, w \in \mathbb{C}$: $\quad |z \cdot w| = |z| \cdot |w|$

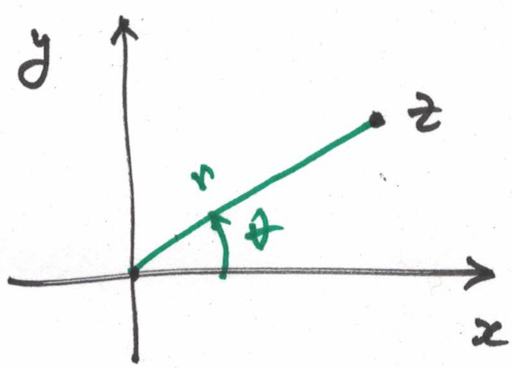
$$z = x + iy = r \cdot e^{i\theta}, \qquad r = |z| = \sqrt{x^2 + y^2}$$

$z \in \mathbb{C}$

Cartesian form. $\qquad$ polar form

$\theta = \arg(z)$: $\cos(\theta) = \frac{x}{r}$

$\sin(\theta) = \frac{y}{r}$



Assume. $(z_n)_{n\geq 1}$, $z_n = x_n + iy_n$, is Cauchy in $\mathbb{C}$.

Given $\varepsilon > 0$ $\exists N$ $\forall n, m > N$, $\underbrace{|z_n - z_m| < \varepsilon}$.

$$|a+ib| = \sqrt{a^2+b^2} \geq \sqrt{a^2} = |a| \qquad |Re(z)| \leq |z|$$

$$|a+ib| = \sqrt{a^2+b^2} \geq \sqrt{b^2} = |b|. \qquad |Im(z)| \leq |z|.$$

$$\underbrace{|Re(z_n - z_m)|}_{|x_n - x_m|} \leq |z_n - z_m| < \varepsilon. \to |x_n - x_m| < \varepsilon$$

$$\underbrace{|Im(z_n - z_m)|}_{y_n - y_m} \leq |z_n - z_m| < \varepsilon \to |y_n - y_m| < \varepsilon.$$

$\Rightarrow$ $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ are Cauchy in $\mathbb{R}$.

$\to$ Convergent $\to$ $\lim_{n\to\infty} x_n = u$, $\lim_{n\to\infty} y_n = v$

$u, v \in \mathbb{R}$

$$|z_n - (u+iv)| \leq |x_n - u| + |y_n - v|. \Rightarrow \boxed{\lim_{n\to\infty} z_n = u+iv}$$

$\|$

$x_n + iy_n$

$\downarrow$

$(z_n)_{n\geq 1}$ is Convergent in $\mathbb{C}$

$$|(x_n - u) + i(y_n - v)| \leq |x_n - u| + |y_n - v|.$$