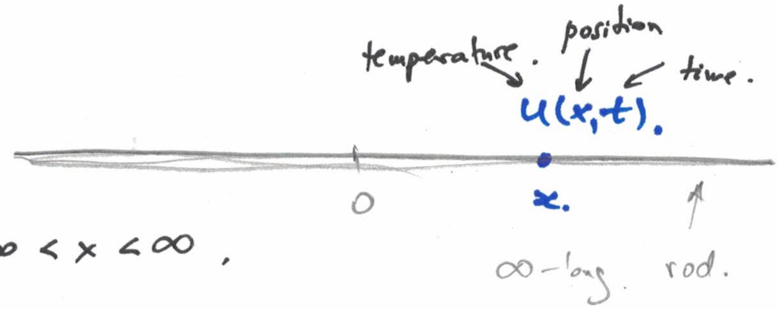
L23

The Heat Equation

1. The Heat Equation on the Real Line

$$\begin{cases} \dfrac{\partial u}{\partial t} = a \cdot \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = \underbrace{\varphi(x)}, \qquad -\infty < x < \infty. \end{cases}$$



initial condition: initial temperature field.

Want: $u = u(x,t)$, $t \geq 0$.

Step 1. Apply the Fourier Transform with respect to x-variable.

Let $\hat{u}(s,t) =$ F.T. of the function $x \longmapsto u(x,t)$.

$$\hat{u}(s,t) = \int_{-\infty}^{\infty} e^{-2\pi i s x}\, u(x,t)\, dx$$

$$u(x,t) = \int_{-\infty}^{\infty} e^{2\pi i s x}\, \hat{u}(s,t)\, ds.$$

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} e^{2\pi i s x}\, \hat{u}(s,t)\, ds = \int_{-\infty}^{\infty} e^{2\pi i s x}\, \frac{\partial \hat{u}}{\partial t}(s,t)\, ds$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\int_{-\infty}^{\infty} e^{2\pi i s x}\, \hat{u}(s,t)\, ds = \int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x} e^{2\pi i s x}\right) \hat{u}(s,t)\, ds = \int_{-\infty}^{\infty} e^{2\pi i s x}\, 2\pi i s \cdot \hat{u}(s,t)\, ds$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\int_{-\infty}^{\infty} e^{2\pi i s x}\, 2\pi i s\, \hat{u}(s,t)\, ds = \int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x} e^{2\pi i s x}\right) 2\pi i s\, \hat{u}(s,t)\, ds = \int_{-\infty}^{\infty} e^{2\pi i s x}\, (2\pi i s)^2\, \hat{u}(s,t)$$

$$\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2}, \text{ all } x,$$

$$\int_{-\infty}^{\infty} e^{2\pi i s x} \frac{\partial \hat{u}}{\partial t}(s,t)\,ds = \int_{-\infty}^{\infty} e^{2\pi i s x}\, a(2\pi i s)^2\, \hat{u}(s,t)\,ds.$$

$$\Rightarrow \quad \frac{\partial \hat{u}}{\partial t}(s,t) = a\underbrace{(2\pi i s)^2}_{-4\pi^2 s^2}\hat{u}(s,t), \quad \text{all } s, \ -\infty < s < \infty$$

$$\frac{\partial \hat{u}}{\partial t} + 4\pi^2 a s^2 \cdot \hat{u} = 0., \quad \text{all } s, \text{ all } t.$$

Step 2. Fix $s \in \mathbb{R}$. Let $f(t) = \hat{u}(s,t)$,

$$f' + 4\pi^2 a s^2 \cdot f = 0.$$

$$\frac{df}{dt} + 4\pi^2 a s^2 \cdot f = 0.$$

$$r + 4\pi^2 a s^2 = 0 \longrightarrow r = -4\pi^2 a s^2$$

$$f(t) = C \cdot e^{rt} = C e^{-4\pi^2 a s^2 t}$$

C is independent of t, but C may depend on s: $C = C(s)$

$$\Rightarrow \quad \hat{u}(s,t) = C(s) \cdot e^{-4\pi^2 a s^2 t}$$

Initial condition: $u(x,0) = \varphi(x)$. $\Rightarrow$ $\hat{u}(s,0) = \int_{-\infty}^{\infty} e^{-2\pi i s x} \varphi(x)\,dx.$

$$t=0 \Rightarrow \hat{u}(s,0) = C(s) \cdot 1.$$

$$C(s) = \hat{u}(s,0) = \int_{-\infty}^{\infty} e^{-2\pi i s x} \varphi(x)\,dx = \hat{\varphi}(s).$$

$$\boxed{\hat{u}(s,t) = \hat{\varphi}(s) \cdot e^{-4\pi^2 a s^2 t}}$$

Step 3. Invert Fourier transform:

$$u(x,t) = \int_{-\infty}^{\infty} e^{2\pi i s x} \underbrace{e^{-4\pi^2 a s^2 t}} \cdot \hat{\varphi}(s)\, ds.$$

$$= \left(\underline{\text{inverse F.T. of } s \mapsto e^{-4\pi^2 a s^2 t}} \right) * \varphi.$$

$$e^{-\gamma \cdot s^2}, \quad \underline{\gamma = 4\pi^2 a t}$$

$f(x)$	$F(s)$
$e^{-\pi x^2}$	$e^{-\pi s^2}$

$$\underbrace{\sqrt{\frac{\pi}{\gamma}} \cdot e^{-\pi\left(\sqrt{\frac{\pi}{\gamma}} \cdot x\right)^2}} \leftarrow \cdots e^{-\gamma \cdot s^2} = e^{-\pi \cdot \left(\sqrt{\frac{\gamma}{\pi}} s\right)^2}$$

$$\sqrt{\frac{\pi}{4\pi^2 a t}} \cdot e^{-\frac{\pi^2}{4\pi^2 a t} x^2} = \frac{1}{\sqrt{4\pi a t}} e^{-\frac{x^2}{4at}}$$

let $$K_t(x) = \frac{1}{\sqrt{4\pi a t}} e^{-\frac{x^2}{4at}} \quad \text{(Heat kernel).}$$

Its Fourier transform: is $e^{-4\pi^2 a t s^2}$

$$\Rightarrow \quad u(x,t) = (K_t * \varphi)(x) = \int_{-\infty}^{\infty} k_t(x-y)\, \varphi(y)\, dy =$$

$$= \frac{1}{\sqrt{4\pi a t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4at}} \cdot \varphi(y)\, dy.$$

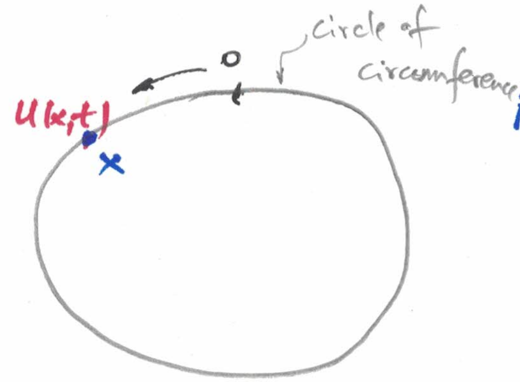
As $t \downarrow 0$ (approaches 0 from above):

Assume $\varphi \in \mathcal{S}$.

$$\lim_{t \downarrow 0} u(x,t) = \lim_{t \downarrow 0} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi a t}} e^{-\frac{(x-y)^2}{4at}} \cdot \varphi(y)\,dy =$$

$$= \int \delta(x-y)\,\varphi(y)\,dy = \varphi(x) = u(x,0).$$

Heat Equation on $\mathbb{T}^1$



$$\begin{cases} \dfrac{\partial u}{\partial t} = a \dfrac{\partial^2 u}{\partial x^2} & \leftarrow \text{Diff. Equation.} \\ u(x,0) = \varphi(x). & \leftarrow \text{Initial Condition} \\ u(x+p,t) = u(x,t). & \leftarrow \text{Periodic Solutions} \end{cases}$$

Step 1. Expand. $u(x,t) = \sum_n c_n(t)\, e^{\frac{2\pi i n x}{p}}$

in a Fourier Series.

$$c_n(t) = \frac{1}{p}\int_0^p e^{-\frac{2\pi i n x}{p}}\, u(x,t)\,dx \quad : n^{th} \text{ Fourier coefficient.}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\sum_n c_n(t)\, e^{\frac{2\pi i n x}{p}} = \sum_{n=-\infty}^{\infty} \left(\frac{d c_n(t)}{dt}\right) \cdot e^{\frac{2\pi i n x}{p}}.$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\Big(\sum_{n} c_n(t)\, \underbrace{e^{\frac{2\pi i n x}{p}}}\Big) = \sum_{n=-\infty}^{\infty} c_n(t) \cdot \frac{2\pi i n}{p}\, e^{\frac{2\pi i n x}{p}}$$

depends on x

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\Big(\sum_{n} c_n(t)\, \frac{2\pi i n}{p}\, e^{\frac{2\pi i n x}{p}}\Big) = \sum_{n=-\infty}^{\infty} c_n(t) \cdot \left(\frac{2\pi i n}{p}\right)^2 e^{\frac{2\pi i n x}{p}}$$

$$\frac{\partial u}{\partial t} = a\,\frac{\partial^2 u}{\partial x^2} \Rightarrow \sum_{n} \frac{dc_n(t)}{dt} \cdot e^{\frac{2\pi i n x}{p}} = \sum_{n} a\left(\frac{2\pi i n}{p}\right)^2 \cdot c_n(t)\, e^{\frac{2\pi i n x}{p}}$$

$$\Rightarrow \quad \frac{dc_n}{dt} = a\left(\frac{2\pi i n}{p}\right)^2 \cdot c_n$$

$$\underbrace{\frac{dc_n}{dt} + \frac{4\pi^2 a n^2}{p^2} \cdot c_n = 0.}_{}$$

, for every integer $n \in \mathbb{Z}$
for all $t \geq 0$.

Characteristic Eqn.:

$$r + \frac{4\pi^2 a n^2}{p^2} = 0 \rightarrow r = -\frac{4\pi^2 a n^2}{p^2}$$

Step 2: Solve the ODE.

$$c_n(t) = c_n(0) \cdot e^{rt} = c_n(0)\, e^{-\frac{4\pi^2 a n^2}{p^2} t}$$

Initial Condition:

$$\varphi(x) = u(x,0) = \sum_{n} c_n(0)\, e^{\frac{2\pi i n x}{p}}.$$

$$\rightarrow \underbrace{c_n(0) = \frac{1}{p}\int_0^p e^{-\frac{2\pi i n x}{p}}\, \varphi(x)\, dx}_{}.$$

Step 3.

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n(t)\, e^{\frac{2\pi i n x}{p}} = \sum_{n=-\infty}^{\infty} c_n(0)\, e^{-\frac{4\pi^2 a n^2}{p^2} t}\, e^{\frac{2\pi i n x}{p}}$$

$$= \sum_n c_n(0)\, e^{-\frac{4\pi^2 a n^2}{p^2} t + \frac{2\pi i n}{p} \cdot x}$$

$$= \sum_{n} \underbrace{\frac{1}{p} \int_0^p e^{-\frac{2\pi i n y}{p}} \varphi(y)\, dy}_{c_n(0)} .\; e^{\frac{2\pi i n}{p} x - \frac{4\pi^2 a n^2}{p^2} t} =$$

$$= \int_0^p \underbrace{\frac{1}{p} \sum_{n=-\infty}^{\infty} e^{\frac{2\pi i n}{p}(x-y) - \frac{4\pi^2 a n^2}{p^2} t}}_{H(t,\, x-y)} \cdot \varphi(y)\, dy \; . =$$

$H(t, x-y)$. : discrete heat kernel at time t.

$$= \int_0^p H(t, x-y)\, \varphi(y)\, dy = \Big(H(t, \cdot) \underset{\uparrow}{*} \varphi \Big)(x).$$

p-periodic convolution.

$$\lim_{t \to 0} u(x,t) = \lim_{t \to 0} \sum_n c_n(0) \underbrace{e^{-\frac{4\pi^2 a n^2}{p^2} t}}_{\downarrow \; 1} e^{\frac{2\pi i n}{p} x} = \sum_n c_n(0)\, e^{\frac{2\pi i n x}{p}} = \varphi(x).$$

$$\lim_{t \to \infty} u(x,t) = c_0(0) = \frac{1}{p} \int_0^p \varphi(x)\, dx \equiv \text{average temperature}.$$