The Wave Equation

The Wave Equation on $\mathbb{R}$.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & -\infty < x < \infty. \\ u(x,0) = \varphi_0(x). & \\ \dfrac{\partial u}{\partial t}(x,0) = \varphi_1(x). & \end{cases}$$

$c > 0$ the wave speed

Solution.

Start: $\hat{u}(s,t) = \int_{-\infty}^{\infty} e^{-2\pi i x s}\, u(x,t)\,dx$, $u(x,t) = \int_{-\infty}^{\infty} e^{2\pi i x s}\, \hat{u}(s,t)\,ds.$

$$\frac{\partial^2 u}{\partial t^2} = \int_{-\infty}^{\infty} e^{2\pi i x s} \frac{\partial^2 \hat{u}}{\partial t^2}\, ds.$$

$$\frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} e^{2\pi i x s} (2\pi i s)^2\, \hat{u}(s,t)\,ds.$$

$$u_{tt} = c^2 u_{xx} \Rightarrow \frac{\partial^2 \hat{u}}{\partial t^2} = c^2 \cdot (-4\pi^2 s^2)\,\hat{u}$$

let $f(t) = \hat{u}(s,t)$ for a fixed s.

$$f'' + 4\pi^2 c^2 s^2 \cdot f = 0.$$

characteristic equation: $r^2 + 4\pi^2 c^2 s^2 = 0.$

$$r_{1,2} = \pm 2\pi i c s.$$

$$f(t) = A e^{2\pi i s c t} + B e^{-2\pi i s c t}$$

$$\hat{u}(s,t) = A(s) \cdot e^{2\pi i s c t} + B(s) e^{-2\pi i s c t}$$

Initial condition $\Rightarrow$ $A(s), B(s)$.

$$u(x,t) = \int_{-\infty}^{\infty} e^{2\pi i s x} \hat{u}(s,t) ds = \int_{-\infty}^{\infty} e^{2\pi i s x} e^{2\pi i s c t} A(s) ds +$$

$$+ \int_{-\infty}^{\infty} e^{2\pi i s x} e^{-2\pi i s c t} B(s) ds = \int_{-\infty}^{\infty} e^{2\pi i s (x+ct)} A(s) ds +$$

$$+ \int_{-\infty}^{\infty} e^{2\pi i s (x-ct)} B(s) ds = a(x+ct) + b(x-ct).$$

where $a(\cdot)$ is the inverse Fourier transform of $A(s)$.
$b(\cdot)$ is the inverse Fourier transform of B.

$$u(x,t) = a(x+ct) + b(x-ct).$$

Let's find $a(\cdot), b(\cdot)$ (functions a and b).

$t=0$: $\quad \varphi_0(x) = u(x,0) = a(x) + b(x).$

$$\varphi_1(x) = \frac{\partial u}{\partial t}(x,t)\Big|_{t=0} = a'(x+c\cdot 0)\cdot c + b'(x-c\cdot 0)\cdot(-c)$$

$$= c \cdot a'(x) - c \cdot b'(x).$$

$$\left\{\begin{array}{l} a(x)+b(x) = \varphi_0(x). \\ a'(x) - b'(x) = \frac{1}{c}\varphi_1(x). \end{array}\right. \xrightarrow[\text{w.r.t. } x]{\text{derivative}} \left.\begin{array}{l} a'(x)+b'(x) = \varphi_0'(x). \\ a'(x) - b'(x) = \frac{1}{c}\varphi_1(x) \end{array}\right\}$$

Sum :
$$2a'(x) = \varphi_0'(x) + \frac{1}{c}\varphi_1(x).$$
$$a'(x) = \frac{1}{2}\varphi_0'(x) + \frac{1}{2c}\varphi_1(x).$$

Integrate w.r.t. x

$$\begin{cases} a(x) = \frac{1}{2}\varphi_0(x) + \frac{1}{2c}\int_0^x \varphi_1(y)dy + C_1. \\ b(x) = \varphi_0(x) - a(x) = \frac{1}{2}\varphi_0(x) - \frac{1}{2c}\int_0^x \varphi_1(y)dy - C_1. \end{cases}$$

$$u(x,t) = a(x+ct) + b(x-ct) = \frac{1}{2}\varphi_0(x+ct) + \frac{1}{2}\varphi_0(x-ct) +$$
$$+ \frac{1}{2c}\int_0^{x+ct}\varphi_1(y)dy \underbrace{- \frac{1}{2c}\int_0^{x-ct}\varphi_1(y)dy}_{+\frac{1}{2c}\int_{x-ct}^{0}\varphi_1(y)dy.} + \underbrace{C_1 - C_1}_{0.} =$$

$$= \frac{1}{2}\varphi_0(x+ct) + \frac{1}{2}\varphi_0(x-ct) + \frac{1}{2c}\Bigg[\underbrace{\int_{x-ct}^{0}\varphi_1(y)dy + \int_0^{x+ct}\varphi_1(y)dy}_{\int_{x-ct}^{x+ct}\varphi_1(y)\,dy.}\Bigg].$$

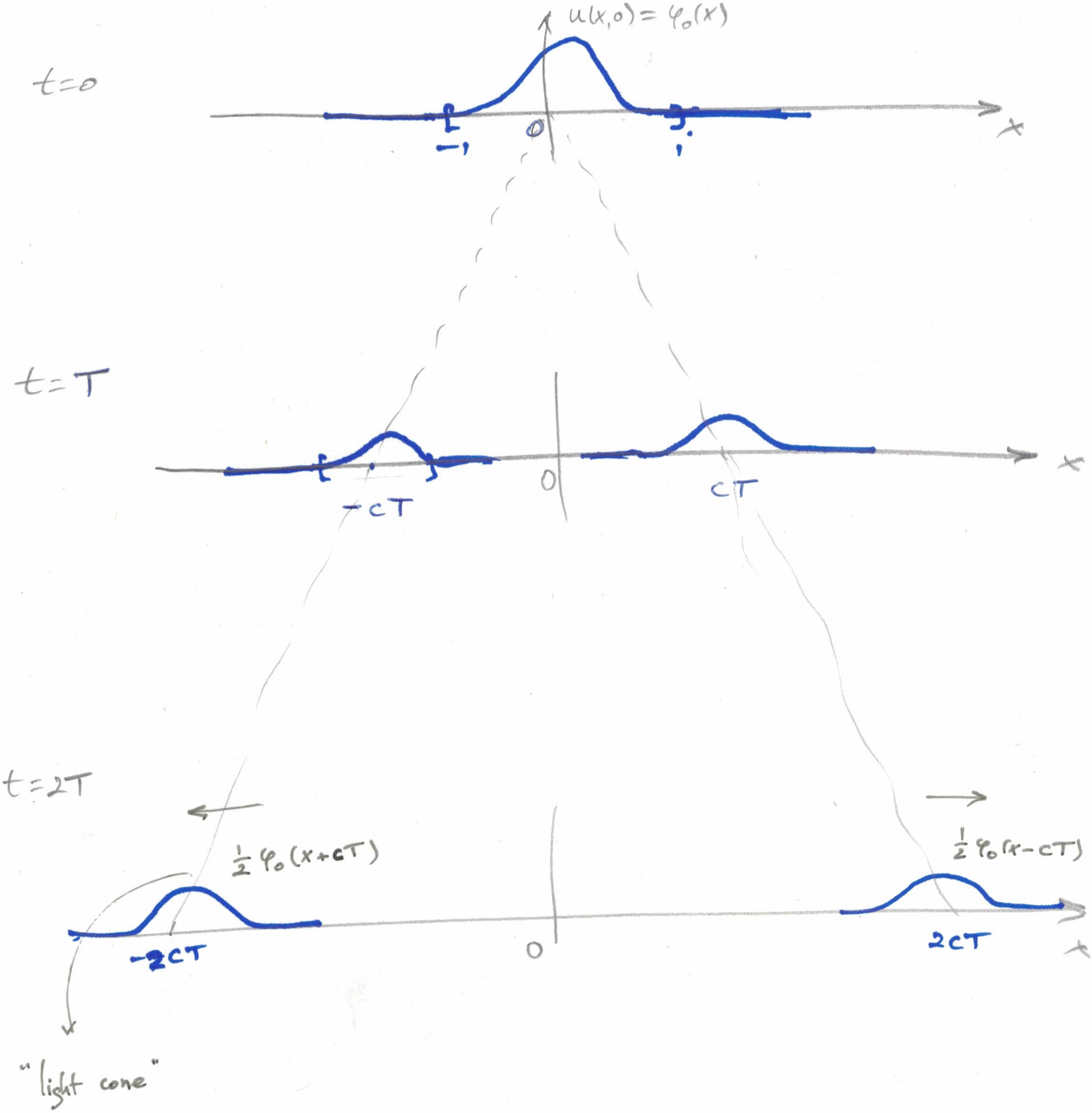
Solution is :

$$\boxed{u(x,t) = \frac{1}{2}\varphi_0(x+ct) + \frac{1}{2}\varphi_0(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct}\varphi_1(y)dy}$$

Remark:

Assume $\varphi_1(x) = 0 = \frac{\partial u}{\partial t}(x,0)$.

$$u(x,t) = \frac{1}{2}\varphi_0(x+ct) + \frac{1}{2}\varphi_0(x-ct)$$



The Wave Equation with Periodic Boundary Condition (on $\mathbb{T}^1$)

$$\left\{\begin{array}{ll} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & \text{diff. equation.} \\ u(x,0) = \varphi_0(x) & \\ \dfrac{\partial u}{\partial t}(x,0) = \varphi_1(x). & \Big\} \text{ initial conditions} \\ u(x+p,t) = u(x,t). & \rightarrow p\text{-periodic solutions} \end{array}\right.$$

Method 1. $\displaystyle u(x,t) = \sum_{n=-\infty}^{\infty} c_n(t)\, e^{\frac{2\pi i n x}{p}}$

Method 2.

$$u(x,t) = \frac{1}{2}\varphi_0(x+ct) + \frac{1}{2}\varphi_0(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \varphi_1(y)\,dy.$$

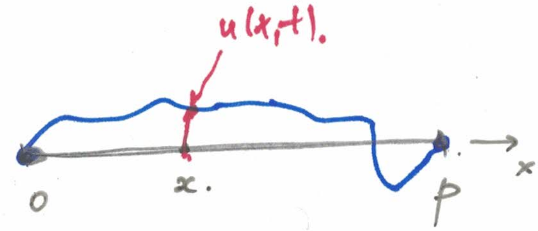
If φ_0, φ_1 are p-periodic.

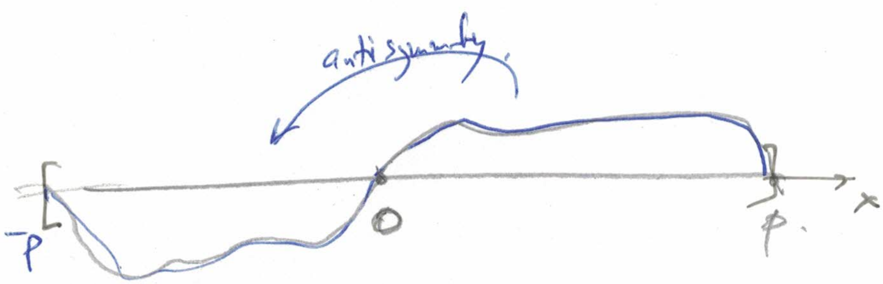
$$\Rightarrow \underline{u(x+p,t) = u(x,t).}$$

Vibrating String with Fixed Endpoints

Want: $u(x,t)$.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}, & 0 \leq x \leq p. \\ u(x,0) = f(x), & 0 < x < p \\ \dfrac{\partial u}{\partial t}(x,0) = g(x), & 0 < x < p \\ u(0,t) = 0, & \text{for all } t. \\ u(p,t) = 0, & \text{for all } t. \end{cases}$$





Extend $u(x,t)$ from $0 < x < p$ to $-p < x < 0$ by:

$$\underline{\underline{u(-x,t) = -u(x,t).}}$$

Expand the antisymmetric extension in sine-cosine functions:

At every t.

$$u(x,t) = C_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos\left(\frac{2\pi n x}{2p}\right) + \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{2\pi n x}{2p}\right)$$

$$C_0(t) = \frac{1}{2p} \int_{-p}^{p} \overset{\text{extension}}{u(x,t)}\, dx. = 0.$$

$$a_n(t) = \frac{2}{2p} \int_{-p}^{p} \underbrace{\cos\left(\frac{2\pi n x}{2p}\right)}_{\text{sym.}} \cdot \underbrace{u(x,t)}_{\text{antisym.}}\, dx = 0.$$

$$\Rightarrow \quad u(x,t) = \sum_{n\geq 1}^{\infty} b_n(t) \cdot \sin\left(\frac{\pi n x}{p}\right). \qquad (*)$$

where
$$b_n(t) = \frac{2}{2p} \int_{-p}^{p} \underbrace{\underbrace{\sin\left(\frac{\pi n x}{p}\right)}_{\text{antisym.}} \cdot \underbrace{u(x,t)}_{\text{antisymm.}}}_{\text{Symmetric in } x} dx = \underline{\frac{2}{p} \int_{0}^{p} \sin\left(\frac{\pi n x}{p}\right) u(x,t)\, dx.}$$

Substitute $(*)$ into $u_{tt} = c^2 u_{xx}$

$$\sum_{n\geq 1}^{\infty} \frac{d^2 b_n(t)}{dt^2} \cdot \sin\left(\frac{\pi n x}{p}\right) = -c^2 \sum_{n\geq 1}^{\infty} b_n(t) \cdot \left(\frac{\pi n}{p}\right)^2 \sin\left(\frac{\pi n x}{p}\right).$$

$$b_n'' = -\left(\frac{\pi n c}{p}\right)^2 \cdot b_n$$

$$b_n'' + \left(\frac{\pi n c}{p}\right)^2 b_n = 0.$$

General solution : $r^2 + \left(\frac{\pi n c}{p}\right)^2 = 0 \longrightarrow r_{1,2} = \pm i \frac{\pi n c}{p}$.

$$b_n(t) = \alpha_n \cdot \cos\left(\frac{\pi n c t}{p}\right) + \beta_n \sin\left(\frac{\pi n c t}{p}\right), \quad n \geq 1.$$

$$\underline{u(x,t) = \sum_{n\geq 1}^{\infty} \left[\alpha_n \cos\left(\frac{\pi n c t}{p}\right) + \beta_n \sin\left(\frac{\pi n c t}{p}\right)\right] \sin\left(\frac{\pi n x}{p}\right).}$$

Initial condition: $t = 0$.

$$\begin{cases} f(x) = u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{\pi n x}{p}\right) & \dashrightarrow \text{Sine expansion of } f \\ g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \beta_n \cdot \frac{\pi n c}{p} \sin\left(\frac{\pi n x}{p}\right). & \dashrightarrow \text{perform a sine expansion of } g \end{cases}$$

$$\Rightarrow \boxed{\alpha_n = \frac{2}{p}\int_0^p \sin\left(\frac{\pi n x}{p}\right) f(x)\, dx}$$

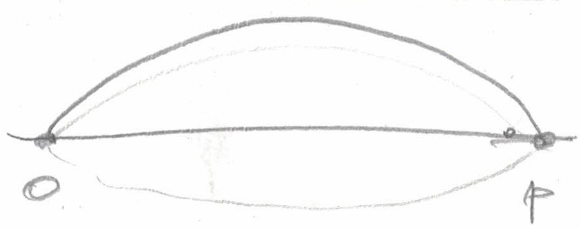
$$\beta_n \frac{\pi n c}{p} = \frac{2}{p}\int_0^p \sin\left(\frac{\pi n x}{p}\right) g(x)\, dx \Rightarrow \boxed{\beta_n = \frac{2}{\pi n c}\int_0^p \sin\left(\frac{\pi n x}{p}\right) g(x)\, dx}$$

Remarks:

$n = 1$:

$$u_1(x,t) = \left(\alpha_1 \cos\left(\frac{\pi c t}{p}\right) + \beta_1 \sin\left(\frac{\pi c t}{p}\right)\right) \sin\left(\frac{\pi x}{p}\right) =$$

$$= \sqrt{\alpha_1^2 + \beta_1^2} \cos\left(\frac{\pi c t}{p} + \varphi_1\right) \cdot \sin\left(\frac{\pi x}{p}\right)$$



Oscillates over time: Time period: $\frac{\pi c \cdot T_1}{p} = 2\pi$

$$\boxed{T_1 = \frac{2p}{c}} \dashrightarrow \text{Frequency of oscillation} \quad \boxed{f_1 = \frac{1}{T_1} = \frac{c}{2p}}$$

Fundamental frequency (pitch frequency)