Solution of the Wave Equation.

$$u_{tt} = c^2 \cdot u_{xx}, \qquad u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x).$$

1. Let $\hat{u}(s,t)$ denote the Fourier transform of $u(x,t)$ w.r.t. x-variable:

$$\hat{u}(s,t) = \int_{-\infty}^{\infty} e^{-2\pi i s x}\, u(x,t)\, dx, \qquad u(x,t) = \int_{-\infty}^{\infty} e^{2\pi i s x}\, \hat{u}(s,t)\, ds$$

$$u_{tt} = \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} e^{2\pi i s x}\, \hat{u}(s,t)\, ds = \int_{-\infty}^{\infty} e^{2\pi i s x} \cdot \frac{\partial^2 \hat{u}}{\partial t^2}(s,t)\, ds.$$

$$\parallel$$

$$c^2 u_{xx} = c^2 \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} e^{2\pi i s x}\, \hat{u}(s,t)\, ds = \int_{-\infty}^{\infty} c^2 (2\pi i s)^2 \cdot \hat{u}(s,t)\, e^{2\pi i s x}\, ds.$$

$$\Rightarrow \quad \frac{\partial^2 \hat{u}}{\partial t^2}(s,t) = (2\pi i s c)^2 \cdot \hat{u}(s,t), \quad \text{for all } s \in \mathbb{R}, \text{ for all } t.$$

2. Solve for $\hat{u}$:

<u>Fix $s \in \mathbb{R}$.</u> Let $f(t) = \hat{u}(s,t)$. $\Rightarrow \frac{\partial^2 \hat{u}}{\partial t^2} = f''$

$$f'' = -(2\pi s c)^2 \cdot f$$

$$f'' + (2\pi s c)^2 \cdot f = 0$$

Charact. Equation: $r^2 + (2\pi s c)^2 = 0 \Rightarrow r_{1,2} = \pm 2\pi i s c.$

$$f(t) = A e^{2\pi i s c t} + B e^{-2\pi i s c t}$$

where $A = A(s)$, $B = B(s)$ may depend on s.

We shall find $A = A(s)$, $B = B(s)$ in terms of the initial condition.

$$\hat{u}(s,t) = A(s)\,e^{2\pi i s c t} + B(s)\,e^{-2\pi i s c t}$$

$$\Rightarrow \quad u(x,t) = \int_{-\infty}^{\infty} e^{2\pi i s x}\,\hat{u}(s,t)\,ds = \int_{-\infty}^{\infty} e^{2\pi i s x + 2\pi i s c t} \cdot A(s)\,ds +$$

$$+ \int_{-\infty}^{\infty} e^{2\pi i s x - 2\pi i s c t} \cdot B(s)\,ds = a(x+ct) + b(x-ct)$$

where $a(\cdot)$, $b(\cdot)$ are the inverse Fourier transforms of A and B respectively.

3. Initial condition:

$t=0$: $\quad \varphi_0(x) = u(x,0) = a(x) + b(x)$

$t=0$: $\quad \varphi_1(x) = \frac{\partial u}{\partial t}(x,0) = a'(x+ct)\Big|_{t=0} \cdot c + b'(x-ct)\Big|_{t=0} \cdot (-c)$

$$= c \cdot a'(x) - c \cdot b'(x).$$

We obtained:

$$\begin{cases} a(x) + b(x) = \varphi_0(x). \\ a'(x) - b'(x) = \frac{1}{c}\varphi_1(x). \end{cases} \dashrightarrow \quad \begin{array}{l} a'(x) + b'(x) = \varphi_0'(x) \\ a'(x) - b'(x) = \frac{1}{c}\varphi_1(x). \\ \hline \end{array}$$

$+$: $\quad 2a'(x) = \varphi_0'(x) + \frac{1}{c}\varphi_1(x).$

$-$: $\quad 2b'(x) = \varphi_0'(x) - \frac{1}{c}\varphi_1(x)$

$$\Rightarrow \begin{cases} a'(x) = \frac{1}{2}\varphi_0'(x) + \frac{1}{2c}\varphi_1(x) \\ b'(x) = \frac{1}{2}\varphi_0'(x) - \frac{1}{2c}\varphi_1(x). \end{cases} \xrightarrow[\text{w.r.t. } x]{\text{Integrate.}} \begin{cases} a(x) = \frac{1}{2}\varphi_0(x) + \frac{1}{2c}\int_0^x \varphi_1(y)\,dy + C_1 \\ b(x) = \frac{1}{2}\varphi_0(x) - \frac{1}{2c}\int_0^x \varphi_1(y)\,dy + C_2 \end{cases}$$

$$\varphi_0(x) = a(x) + b(x) = \varphi_0(x) + \cancel{\frac{1}{2c}\int_0^x \varphi_1(y)\,dy} - \cancel{\frac{1}{2c}\int_0^x \varphi_1(y)\,dy} + C_1 + C_2 \Rightarrow \underline{C_1 + C_2 = 0}$$

Thus:

$$u(x,t) = a(x+ct) + b(x-ct) = \frac{1}{2}\varphi_0(x+ct) + \frac{1}{2c}\int_0^{x+ct}\varphi_1(y)\,dy + \underline{C_1} +$$

$$+ \frac{1}{2}\varphi_0(x-ct) - \frac{1}{2c}\int_0^{x-ct}\varphi_1(y)\,dy + \underline{C_2} =$$

$$= \frac{1}{2}\Big(\varphi_0(x+ct) + \varphi_0(x-ct)\Big) + \frac{1}{2c}\left[\int_0^{x+ct}\varphi_1(y)\,dy + \int_{x-ct}^{0}\varphi_1(y)\,dy\right] =$$

$$= \frac{1}{2}\Big(\varphi_0(x+ct) + \varphi_0(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct}\varphi_1(y)\,dy$$

<u>Wave Equation on $\mathbb{T}^1$</u> (with Periodic Boundary Conditions).:

$$\left\{\begin{array}{l} u_{tt} = c^2 u_{xx} \\ u(x,0) = \varphi_0(x) \\ u_t(x,0) = \varphi_1(x) \\ u(x+p,t) = u(x,t) \end{array}\right\} \Rightarrow u(x,t) = \frac{1}{2}\Big(\varphi_0(x+ct) + \varphi_0(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct}\varphi_1(y)\,dy.$$

<u>Q</u>: Is $u(x,t)$ p-periodic in variable x?

<u>Answer</u>: Yes, as long as φ_0 and φ_1 are p-periodic:

$$u(x+p,t) = \frac{1}{2}\Big(\varphi_0(x+p+ct) + \varphi_0(x+p-ct)\Big) + \frac{1}{2c}\int_{x+p-ct}^{x+p+ct}\varphi_1(y)\,dy = \qquad y = u+p.$$

$$= \frac{1}{2}\Big(\underbrace{\varphi_0(x+ct)}_{\text{because } \varphi_0 \text{ is } p\text{-periodic}} + \varphi_0(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct}\underbrace{\varphi_1(u+p)}_{=\varphi_1(u)\text{, because } \varphi_1 \text{ is } p\text{-peri[odic]}}\,du = u(x,t).$$

Alternative Approach:

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n(t) \cdot e^{\frac{2\pi i n x}{p}}$$

, since $x \longmapsto u(x,t)$ is p-periodic, $\forall t$.

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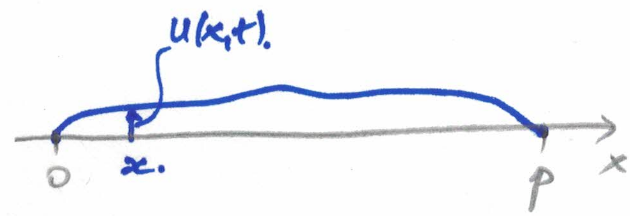
Vibrating String with Fixed End Point

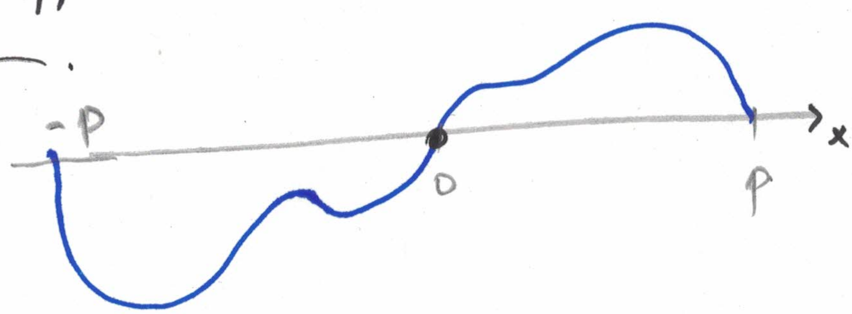


$u(x,t)$ satisfies:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{(PDE).}$$

$u(x,0) = f(x)$
$\frac{\partial u}{\partial t}(x,0) = g(x)$, } Initial Conditions.

$u(0,t) = 0$
$u(p,t) = 0$. } Fixed End Points Condition (Boundary Conditions).

① 

Given $u(x,t)$ defined on $[0,p] \longrightarrow$ Extend to $[-p,0]$ by antisymmetry.

$$u(-x,t) = -u(x,t), \quad x \in [0,p].$$

We obtained an antisymmetric function $u(\cdot,t) : [-p,p] \longrightarrow \mathbb{R}$.
$$x \longmapsto u(x,t).$$

② Expand $u:[-p,p]\to\mathbb{R}$. in Fourier series. w.r.t. variable x.

$$u(x,t)=\sum_{n=-\infty}^{\infty} c_n(t)\cdot e^{\frac{2\pi i n x}{2p}} = \underset{a_0(t)}{\overset{}{\underset{\|}{c_0(t)}}} + \sum_{n=1}^{\infty} a_n(t)\cdot\cos\left(\frac{\pi n x}{p}\right) + \sum_{n=1}^{\infty} b_n(t)\cdot\sin\left(\frac{\pi n x}{p}\right)$$

where: $$c_n(t)=\frac{1}{2p}\int_{-p}^{p} e^{-\frac{2\pi i n x}{2p}}\cdot u(x,t)\,dx.$$

where: $$a_0(t)=c_0(t)=\frac{1}{2p}\int_{-p}^{p} u(x,t)\,dx = 0.$$

$$a_n(t)=\frac{2}{2p}\int_{-p}^{p}\cos\left(\frac{\pi n x}{p}\right)\cdot u(x,t)\,dx. = 0$$

Because: $\cos\left(-\frac{\pi n x}{p}\right)\cdot u(-x,t) = -\cos\left(\frac{\pi n x}{p}\right)\cdot u(x,t).$

$$b_n(t)=\frac{2}{2p}\int_{-p}^{p}\underbrace{\sin\left(\frac{\pi n x}{p}\right)u(x,t)}_{\substack{\text{symmetric in } x\\ \text{(even in } x)}}\,dx. = \frac{2}{p}\int_{0}^{p}\sin\left(\frac{\pi n x}{p}\right)u(x,t)\,dx.$$

We obtained:

$$\left[u(x,t)=\sum_{n=1}^{\infty} b_n(t)\cdot\sin\left(\frac{\pi n x}{p}\right)\right],\qquad b_n(t)=\frac{2}{p}\int_{0}^{p}\sin\left(\frac{\pi n x}{p}\right)\cdot u(x,t)\,dx.$$

Note: $u(0,t)=\sum_{n=1}^{\infty} b_n(t)\cdot\sin(0)=0$, $\quad u(p,t)=\sum_{n=1}^{\infty} b_n(t)\cdot\underbrace{\sin(\pi n)}_{=0}=0.$

Sine-Expansion (as opposed to Sine-Cosine Expansion ≡ Fourier Series).

③ Substitute into Equation:

$$\frac{\partial^2 u}{\partial t^2} = \sum_{n=1}^{\infty} \underbrace{b_n''(t)}_{\shortparallel} \cdot \sin\left(\frac{\pi n x}{p}\right) \qquad \forall x \in [0,p].$$

$$\shortparallel$$

$$c^2 \frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} b_n(t) \cdot c^2 \left(\frac{\pi n}{p}\right)^2 \cdot (-1) \cdot \sin\left(\frac{\pi n x}{p}\right). \qquad \forall t.$$

At every t, For each $n \geq 1$:

$$b_n'' = -\left(\frac{\pi n c}{p}\right)^2 \cdot b_n$$

$$b_n'' + \left(\frac{\pi n c}{p}\right)^2 \cdot b_n = 0$$

General Solution: $K_1 \cdot e^{i\frac{\pi n c t}{p}} + K_2 \, e^{-i\frac{\pi n c t}{p}} \equiv \alpha_n \cdot \cos\left(\frac{\pi n c t}{p}\right) + \beta_n \cdot \sin\left(\frac{\pi n c t}{p}\right)$

For α_n, β_n Scalars.

$$b_n(t) = \alpha_n \cdot \cos\left(\frac{n\pi c t}{p}\right) + \beta_n \cdot \sin\left(\frac{n\pi c t}{p}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty}\left[\alpha_n \cos\left(\frac{n\pi c t}{p}\right) + \beta_n \cdot \sin\left(\frac{n\pi c t}{p}\right)\right] \cdot \sin\left(\frac{n\pi x}{p}\right)$$

④ Use initial condition to solve for α_n, β_n:

At $t=0$:
$$\begin{cases} f(x) = u(x,0) = \displaystyle\sum_{n=1}^{\infty} \alpha_n \cdot \sin\left(\frac{\pi n x}{p}\right) & \longrightarrow \text{Sine expansion of } f \\ g(x) = \dfrac{\partial u}{\partial t}(x,0) = \displaystyle\sum_{n=1}^{\infty} \beta_n \frac{n\pi c}{p} \cdot \sin\left(\frac{\pi n x}{p}\right) & \longrightarrow \text{Sine expansion of } g. \end{cases}$$

Thus:

$$\boxed{\alpha_n = \frac{2}{p}\int_0^p f(x)\cdot \sin\left(\frac{\pi n x}{p}\right)dx}$$

$$\beta_n \cdot \frac{n\pi c}{p} = \frac{2}{p}\int_0^p g(x)\sin\left(\frac{\pi n x}{p}\right)dx. \Rightarrow \boxed{\beta_n = \frac{2}{n\pi c}\int_0^p g(x)\sin\left(\frac{\pi n x}{p}\right)dx}$$

and:

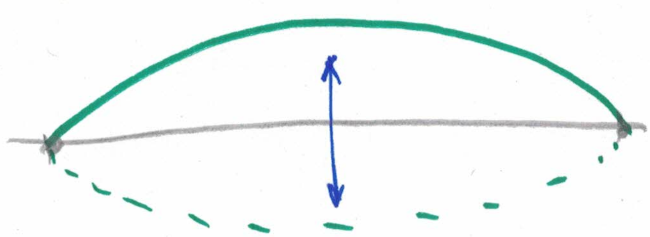
$$\boxed{u(x,t) = \sum_{n\geq 1}^{\infty}\underbrace{\left[\alpha_n \cos\left(\frac{\pi n c t}{p}\right) + \beta_n \cdot \sin\left(\frac{\pi n c t}{p}\right)\right]}_{=\sqrt{\alpha_n^2+\beta_n^2}\,\cos\left(\frac{\pi n c t}{p}+\varphi_n\right).}\sin\left(\frac{n\pi x}{p}\right)}.$$

Remark:

$\boxed{n=1}$

$$u_1(x,t) = \left(\alpha_1 \cos\left(\frac{\pi c t}{p}\right) + \beta_1 \sin\left(\frac{\pi c t}{p}\right)\right)\sin\left(\frac{\pi x}{p}\right) =$$

$$= \sqrt{\alpha_1^2+\beta_1^2}\cos\left(\frac{\pi c t}{p}+\varphi_1\right)\cdot \sin\left(\frac{\pi x}{p}\right) \equiv \left(A_1 \sin\left(\frac{\pi x}{p}\right)\right)\cdot \cos\left(\frac{\pi c t}{p}+\varphi_1\right)$$



Over time, it oscillates.

Time period:

$$\frac{\pi c(t+T_1)}{p} = \frac{\pi c t}{p} + 2\pi$$

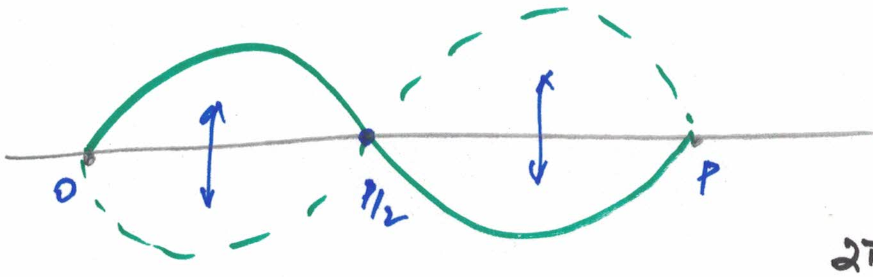
$$\frac{\pi c T_1}{p} = 2\pi \Rightarrow \boxed{T_1 = \frac{2p}{c}}$$

$f_1 = \frac{1}{T_1} = \frac{c}{2p}$: Fundamental frequency (pitch frequency).

$\boxed{n=2}$

$$u_2(x,t)=\left(\alpha_2\cos\left(\frac{2\pi ct}{p}\right)+\beta_2\sin\left(\frac{2\pi ct}{p}\right)\right)\sin\left(\frac{2\pi x}{p}\right)=$$

$$=\left(A_2\cdot\sin\left(\frac{2\pi x}{p}\right)\right)\cdot\cos\left(\frac{2\pi ct}{p}+\varphi_2\right).$$



It oscillates:

$$\frac{2\pi c.T_2}{p}=2\pi$$

$$T_2=\frac{p}{c}=\frac{T_1}{2}\longrightarrow f_2=\frac{1}{T_2}=\frac{2}{T_1}=2f_1$$

⋮

$\boxed{n}$:

$$u_n(x,t)=\left(A_n\sin\left(\frac{n\pi x}{p}\right)\right)\cos\left(\frac{n\pi ct}{p}+\varphi_n\right).$$

It oscillates: $\dfrac{n\pi c\,T_n}{p}=2\pi$

$$T_n=\frac{2p}{nc}\longrightarrow \underbrace{f_n=\frac{1}{T_n}=\frac{nc}{2p}=nf_1}$$

$\boxed{n\text{th mode}}$ $\longrightarrow$ $n\cdot f_1$ oscillating frequency.