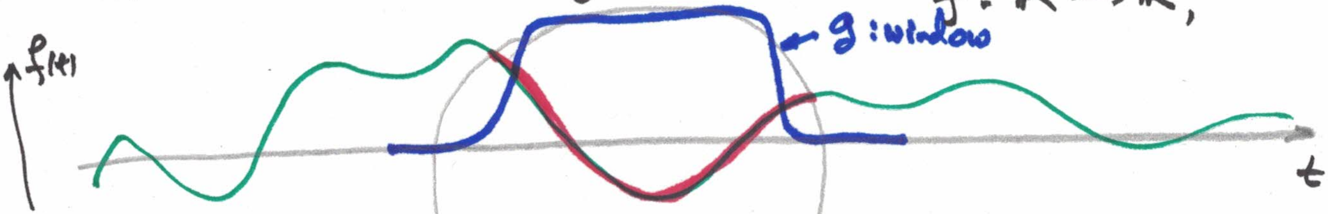
Windowed Fourier Transform (Chapter 11).

Short-Time Fourier Trans

Assume we have a "signal", a function $f: \mathbb{R} \to \mathbb{R}$,



__Want:__ Get "local" information about the frequency content of this signal.

① Frequency may be given by the fourier transform $\hat{f}$.

However the problem with $\hat{f}$ is that to compute it we need $f(t)$ for all t, $-\infty < t < +\infty$.

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\,dt$$

② Even if we know $\hat{f}$, we may want some time-dependent information: e.g., __music__. ... music sheet.

Let $g: \mathbb{R} \to \mathbb{R}$. denote a __window__ function.

The only requirement: $g \in L^2(\mathbb{R})$, i.e., $\int_{-\infty}^{\infty} |g(x)|^2 dx < \infty$.

__Definition.__ The __windowed Fourier transform__ of function $f: \mathbb{R} \to \mathbb{C}$ with respect to window $g: \mathbb{R} \to \mathbb{R}$, is the function $V_g f$,

$$V_g f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}, \qquad V_g f(t, \omega) = \int_{-\infty}^{\infty} e^{-2\pi i \omega x} f(x) \cdot g(x-t)\,dx$$

Properties of the WFT (Windowed Fourier Transform)

$$V_g : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}^2)$$

$$f \longmapsto V_g f$$

signal. its transform (WFT).

a). If. f_1, f_2 are functions, $a_1, a_2 \in \mathbb{C}$, then

$$V_g(a_1 f_1 + a_2 \cdot f_2) = a_1 \cdot V_g(f_1) + a_2 \cdot V_g(f_2)$$

$\Big\}$ V_g is a linear transform

i.e., $V_g(a_1 f_1 + a_2 f_2)(t,\omega) = a_1\ V_g(f_1)(t,\omega) + a_2\ V_g(f_2)(t,\omega)$, all t, ω.

But $\quad V_g(f_1 \cdot f_2) \neq V_g(f_1) \cdot V_g(f_2)$

b) If. f, g_1, g_2 are functions and. $a_1, a_2 \in \mathbb{R}$, then:

$$V_{a_1 g_1 + a_2 g_2}(f) = a_1\ V_{g_1}(f) + a_2\ V_{g_2}(f)$$

c) [Similar to Plancherel]:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |V_g f(t,\omega)|^2\, dt\, d\omega = \int_{-\infty}^{\infty} |f(x)|^2 dx \cdot \int_{-\infty}^{\infty} |g(x)|^2 dx$$

or:

$$\| V_g f \|_{L^2(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R})} \cdot \|g\|_{L^2(\mathbb{R})}$$

(d) [Inversion Formula].

$$f(x) = \frac{1}{\|g\|^2} \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i \omega x} \, V_g f(t,\omega) \cdot g(x-t) \, dt \, d\omega$$

1) works in L^2-sense:

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \left| f(x) - \frac{1}{\|g\|^2} \int_{-R}^{R}\int_{-R}^{R} e^{2\pi i \omega x} \, V_g f(t,\omega) \, g(x-t) \, dt \, d\omega \right|^2 dx = 0.$$

2) works also pointwise, but with. conditions on f and g
Similar to pointwise reconstruction for the Fourier transform.
(see Dirichlet's convergence/pointwise reconstruction result).

(e). [Similar to ~~Plancherel~~ / Parseval identities].

Assume f, f_1, f_2, g_1, g_2 are in $L^2(\mathbb{R})$, g_1, g_2 real valued:

$$\langle V_g f_1, V_g f_2 \rangle_{L^2(\mathbb{R}^2)} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R})} \cdot \|g\|^2$$

$$\langle V_{g_1} f, V_{g_2} f \rangle_{L^2(\mathbb{R}^2)} = \|f\|^2 \cdot \langle g_2, g_1 \rangle.$$

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^2)} = \langle f_1, f_2 \rangle \cdot \langle g_2, g_1 \rangle.$$

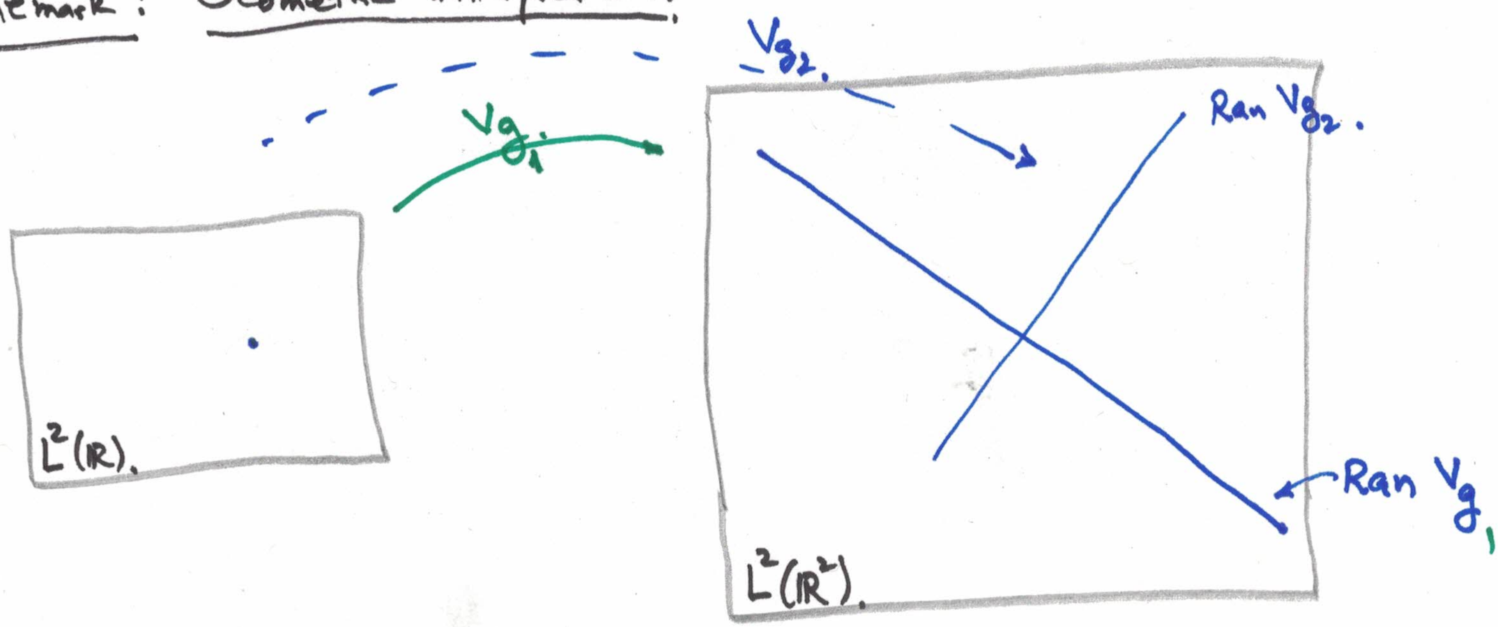
Explicitly:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_g f_1(t,\omega)\cdot\overline{V_g f_2(t,\omega)}\,dt\,d\omega = \left(\int_{-\infty}^{\infty} f_1(x)\cdot\overline{f_2(x)}\,dx\right)\cdot\int_{-\infty}^{\infty}|g(x)|^2\,dx$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_{g_1} f(t,\omega)\,\overline{V_{g_2} f(t,\omega)}\,dt\,d\omega = \int_{-\infty}^{\infty}|f(x)|^2\,dx \cdot \int_{-\infty}^{\infty} g_2(x)\,\overline{g_1(x)}\,dx.$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_{g_1} f_1(t,\omega)\cdot\overline{V_{g_2} f_2(t,\omega)}\,dt\,d\omega = \left(\int_{-\infty}^{\infty} f_1(x)\cdot\overline{f_2(x)}\,dx\right)\cdot\left(\int_{-\infty}^{\infty} g_2(x)\,\overline{g_1(x)}\,dx\right).$$

Remark: Geometric Interpretation.



Assume g_1, g_2 are so that $\int_{-\infty}^{\infty} g_2(x)\,\overline{g_1(x)}\,dx = 0 \iff g_1 \perp g_2.$

$\langle g_2, g_1\rangle = 0.$

Furthermore,

If $\{g_1, g_2, g_3, \dots, g_n, \dots\}$ is ONB for $L^2(\mathbb{R})$

then let $V_{g_n} : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ denote their associated WFT.

let $E_n = \operatorname{Ran} V_{g_n} = \{ V_{g_n} f,\ f \in L^2(\mathbb{R}) \}$.

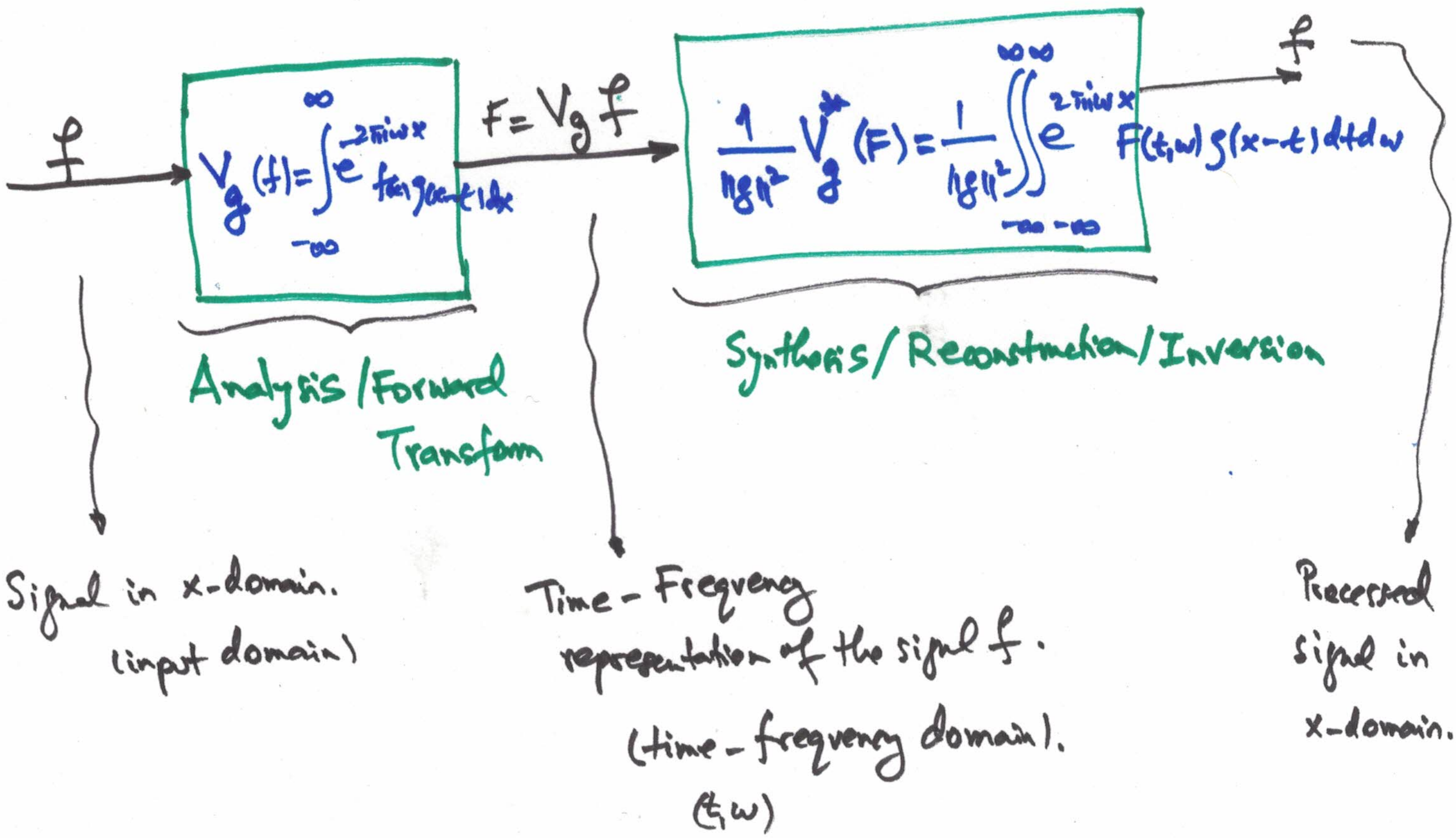
Then:

1) $E_n \perp E_m$, for $n \neq m$

$$\Leftrightarrow \iint F(t,w) \cdot \overline{G(t,w)}\, dt\, dw = 0, \quad F \in E_n,\ G \in E_m.$$

2). $E_1 \oplus E_2 \oplus E_3 \oplus \cdots = L^2(\mathbb{R}^2)$.

<u>Remark</u>. <u>Signal Processing Perspective</u>:



Why the properties (a)-(e) hold?

(a), (b) immediate.

(c): $\|V_g f\|_{L^2(\mathbb{R}^2)} = \|f\| \cdot \|g\|$

$$V_g f(t,\omega) = \int_{-\infty}^{\infty} e^{-2\pi i \omega x} \underbrace{f(x)\cdot g(x-t)}_{\text{donde: } h_t(x)}\, dx. = \int_{-\infty}^{\infty} e^{-2\pi i \omega x} h_t(x)\, dx = \hat{h}_t(\omega).$$

$$\int_{-\infty}^{\infty} |V_g f(t,\omega)|^2\, d\omega = \int_{-\infty}^{\infty} |\hat{h}_t(\omega)|^2\, d\omega \overset{\text{by Plancherel.}}{=} \int_{-\infty}^{\infty} |h_t(x)|^2\, dx$$

$$\|V_g f\|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |V_g f(t,\omega)|^2\, d\omega\, dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h_t(x)|^2\, dx\, dt =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x)|^2 \cdot |g(x-t)|^2\, dx\, dt = \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} |g(x-t)|^2\, dt \right)}_{t \mapsto u = x-t} |f(x)|^2\, dx =$$

$$= \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} |g(u)|^2\, du \right)}_{\text{independent of } x} |f(x)|^2\, dx = \left(\int_{-\infty}^{\infty} |f(x)|^2\, dx \right) \cdot \left(\int_{-\infty}^{\infty} |g(u)|^2\, du \right) =$$

$$= \|f\|^2 \cdot \|g\|^2$$

(e): Follow from (c) by the polarization identity

If f_1, f_2 are real-valued:

$$\underbrace{\langle f_1, f_2 \rangle}_{\text{Scalar product}} = \int_{-\infty}^{\infty} f_1(x) f_2(x)\,dx = \frac{1}{4}\Big[\underbrace{\|f_1+f_2\|^2}_{\text{norms}} - \underbrace{\|f_1-f_2\|^2}_{\text{norms.}}\Big]$$

(d) why inversion as in formula (d):

Since: $f \longmapsto \frac{1}{\|g\|^2} \cdot V_g f$ is an isometry:

$$\|f\|_{L^2(\mathbb{R})} = \Big\| \frac{1}{\|g\|^2} \cdot V_g f \Big\|_{L^2(\mathbb{R}^2)}.$$

Take any function $h \in L^2(\mathbb{R})$,

$$\langle f, h \rangle_{L^2(\mathbb{R})} \overset{\text{by part (e)}}{=\!=} \frac{1}{\|g\|^2} \cdot \langle V_g f, V_g h \rangle_{L^2(\mathbb{R}^2)} = \frac{1}{\|g\|^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_g f(t,\omega) \cdot \overline{\int_{-\infty}^{\infty} e^{-2\pi i \omega x} h(x)\, g(x-t)\,dx}\; dt\,d\omega$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\|g\|^2} V_g f(t,\omega)\, e^{2\pi i \omega x}\, g(x-t)\, \overline{h(x)}\; dt\, d\omega\, dx =$$

$$= \int_{-\infty}^{\infty} \underbrace{\left(\frac{1}{\|g\|^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i \omega x}\, V_g f(t,\omega)\, g(x-t)\, dt\, d\omega\right)}_{\text{function }(x)} \cdot \overline{h(x)}\, dx = \langle \text{function}, h \rangle$$

$\Rightarrow f = \text{function}. \Rightarrow$ Reconstruction formula!

$\square$