Review of Linear Algebra

Vector Space. = Linear Space.

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad x_1, \dots, x_n \in \mathbb{R} \right\}.$$

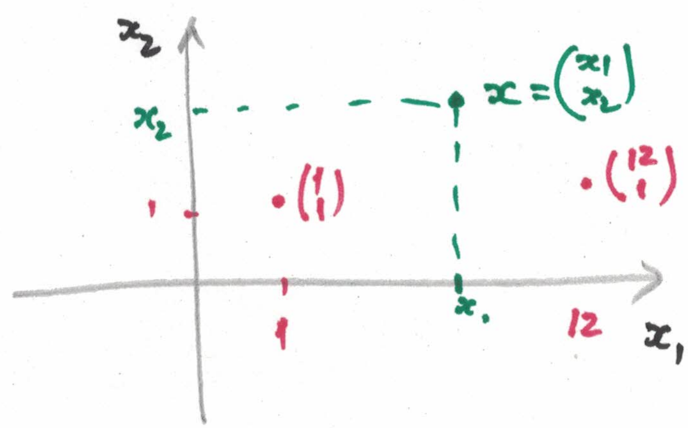
$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : \quad \text{n-dimensional vector.}$$

n = dimension of space $\mathbb{R}^n$

n is a positive integer, $n \geq 1$.

<u>Example</u>.

$n = 2$. , $\mathbb{R}^2$



Addition of vectors.

$$+ : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \; y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \longmapsto x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}.$$

$(\mathbb{R}^n, +)$ is an abelian group.

$$0 \in \mathbb{R}^n \quad , \quad 0\text{-vector}: \quad 0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$-x = \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{pmatrix}.$$

Multiplication with scalars ; <u>Scalar-Vector Multiplication</u>

$$a \in \mathbb{R} \quad , \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \longmapsto a \cdot x = \begin{pmatrix} a x_1 \\ a x_2 \\ \vdots \\ a x_n \end{pmatrix} \in \mathbb{R}^n$$

<u>Convention</u>

$$x \cdot a = a \cdot x = \begin{pmatrix} a x_1 \\ a x_2 \\ \vdots \\ a x_n \end{pmatrix}$$

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field of scalar.

The set $(\mathbb{R}^n, +, \cdot)_{\mathbb{R}}$ is called a <u>(real) vector ~~field~~ space</u>.

scalar-vector multiplication.

<u>Example 2</u>:
$$V = \mathbb{C}^n = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}, \quad z_1 \dots z_n \in \mathbb{C} \right\}.$$

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}, \; w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \longmapsto \underbrace{z + w}_{\text{addition}} = \begin{pmatrix} z_1 + w_1 \\ z_2 + w_2 \\ \vdots \\ z_n + w_n \end{pmatrix} ; \; a \in \mathbb{C} \to \underbrace{a \cdot z}_{\text{scalar-vector multiplication}} = \begin{pmatrix} a z_1 \\ a z_2 \\ \vdots \\ a z_n \end{pmatrix}$$

$(\mathbb{C}^n, +, \cdot)_{\mathbb{C}}$ is a (complex) vector space.

In general:

field. $(F, +, \cdot)$ $\longrightarrow$ For us: $F = \mathbb{R}$

or. $F = \mathbb{C}$

$(V, +, \cdot)_{F}$. is a vector space. (over F).

set of vectors.

1) $(V, +)$: abelian group.

2) $(F, +, \cdot)$: field.

[3). distributivity of $\cdot$ over $+$.].

Example. Fix $1 \leq p < \infty$

$$\ell^p(\mathbb{Z}) = \left\{ x = (x_n)_{n \in \mathbb{Z}} : \sum_{n \in \mathbb{Z}} |x_n|^p < \infty \right\}.$$

$x = (\ldots 0, 0, \underset{\substack{\uparrow \\ 0^{th} \text{ position.}}}{1}, 0, 0, \ldots). \in \ell^p$

But

$y = (\ldots, 1, 1, 1, 1, 1, 1, \underbrace{\ldots}_{\text{constant } 1.}) \notin \ell^p$

If. $x = (x_n)_{n\in\mathbb{Z}}$, $y = (y_n)_{n\in\mathbb{Z}}$ are in $\ell^p(\mathbb{Z})$

then

$\longrightarrow$ $x + y = (x_n + y_n)_{n\in\mathbb{Z}}$ is also in $\ell^p(\mathbb{Z})$.

$a \in \mathbb{C}$, $x = (x_n)_{n\in\mathbb{Z}}$ $\longrightarrow$ $a \cdot x = (a \cdot x_n)_{n\in\mathbb{Z}}$

$(\ell^p(\mathbb{Z}), +, \cdot)_{\mathbb{C}}$ is a complex vector space.

For $p = \infty$.

$$\ell^\infty(\mathbb{Z}) = \left\{ x = (x_n)_{n\in\mathbb{Z}} : \begin{array}{l} \text{there is } M > 0 \text{ s.t.} \\ |x_n| \leq M, \text{ for every } n \end{array} \right\}$$

Consider the case. $V = \mathbb{R}^n$ or $V = \mathbb{C}^n$.

- Linear Independent vectors, sets of linearly independent vectors
- spanning sets
- basis

Linear Independance.

Definition.

A set of vectors $\{v_1, v_2, \dots, v_m\}$ is said linearly independant

if it satisfies:

If. $c_1, c_2, \dots, c_m$ are scalars such that $c_1 v_1 + c_2 v_2 + \dots + c_m v_m = 0$

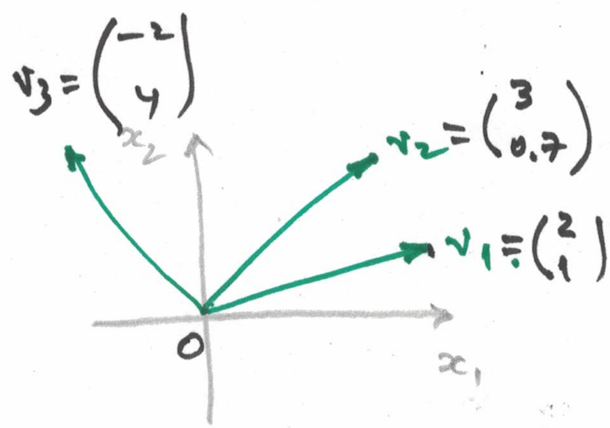
Then $c_1 = c_2 = \dots = c_m = 0$.

In other words:

The only linear combination $c_1 v_1 + c_2 v_2 + \dots + c_m v_m$ that produces the null vector 0 is the combination with all coefficients zero

Examples.

$n=2$,



The set $\{v_1\}$ is linearly independant, as long as $v_1 \neq 0$.

The set $\{v_1, v_2\}$ is linearly independant.

But the set $\{v_1, v_2, v_3\}$ is NOT linearly independant.

Why?: Find $c_2, c_3 \neq 0$ s.t. $v_1 + c_2 v_2 + c_3 v_3 = 0$.

<u>Property</u> If. $\{v_1, v_2, \dots, v_m\}$ is a set of linearly independent vectors in $\mathbb{R}^n$ (or $\mathbb{C}^n$) then $m \leq n$.

Let $\{v_1, v_2, \dots v_m\}$ be a set of vectors in $\mathbb{C}^n$.

$$V = \left(v_1 \middle| v_2 \middle| \dots \middle| v_m \right) \in \mathbb{C}^{n \times m}$$

$\downarrow$

columns of V are the vectors $v_1 \dots v_m$.

<u>Property</u>. The set $\{v_1 \dots v_m\}$ is linearly independent if and only if the only solution of:

$$V \cdot c = 0 \quad , \quad c \in \mathbb{C}^m$$

is the zero vector $c = 0$.

Spanning Sets

Definition A set $\{v_1, v_2, \dots, v_m\} \subset \mathbb{C}^n$ is said **spanning for $\mathbb{C}^n$** if:

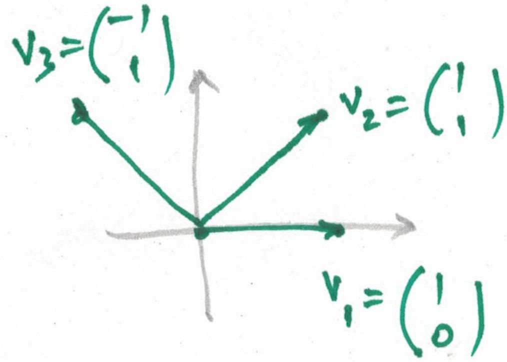
For every $x \in \mathbb{C}^n$ there are $c_1, c_2, \dots, c_m \in \mathbb{C}$ such that

$$c_1 v_1 + c_2 v_2 + \dots + c_m v_m = x.$$

In other words,

Any vector in the space $(\mathbb{C}^n)$ ~~is~~ can be obtained as a linear combination of the spanning set.

Examples. $n=2$



$\{v_1\}$ is NOT a spanning set for $\mathbb{C}^2$.

$\{v_1, v_2\}$ is a spanning set for $\mathbb{C}^2$

$\{v_1, v_2, v_3\}$ is a spanning set for $\mathbb{C}^2$.

Why: If $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ ---> $x = (x_1 - x_2) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 v_1 + c_2 v_2$

where: $c_1 = (x_1 - x_2)$, $c_2 = x_2$.

<u>Property</u> If $\{v_1, v_2, \dots, v_m\}$ is a spanning set for $\mathbb{R}^n$ (or $\mathbb{C}^n$) then $m \geq n$.

Let $V = \left(v_1 \,|\, v_2 \,|\, \dots \,|\, v_m \right) \in \mathbb{C}^{n \times m}$

be the $n \times m$ matrix whose columns are vectors $v_1, v_2, \dots v_m$.

<u>Property</u> The set $\{v_1, \dots, v_m\}$ is spanning in $\mathbb{C}^n$ if and only if, for any $z \in \mathbb{C}^n$ the linear system:

$$z = V \cdot c$$

has a solution $c \in \mathbb{C}^m$.

<u>Fact</u>. Let $\{v_1, v_2, \dots, v_m\} \subset \mathbb{C}^n$. Let $\{w_1, \dots w_n\} \subset \mathbb{C}^m$ be the set of vectors in $\mathbb{C}^m$ such that:

$$\left(v_1 \,|\, v_2 \,|\, \dots \,|\, v_m \right)^{T \,\leftarrow\, \text{transpose}} = \left(w_1 \,|\, \dots \,|\, w_n \right).$$

Then:

$\{v_1, \dots v_m\}$ are linearly independent if and only if $\{w_1, w_2, \dots w_n\}$ is spanning in $\mathbb{C}^m$.

Definition A set $\{v_1, \dots v_m\}$ in $\mathbb{R}^n$ (or $\mathbb{C}^n$) is called a basis if it is simultaneously a linearly independent set and a spanning set.

Property 1. If $\{v_1 \dots v_m\}$ is a basis in $\mathbb{R}^n$ (or $\mathbb{C}^n$) then $m = n$.

Property 2. Let $V = (v_1 | \dots | v_n)$ be the matrix whose columns are the vectors $\{v_1, \dots v_n\}$. Then:

(1) $\{v_1 \dots v_n\}$ is a basis for $\mathbb{R}^n$ (or $\mathbb{C}^n$)

if and only if.

(2) For any $x \in \mathbb{R}^n$ (or $\mathbb{C}^n$) the linear system

$$x = V \cdot c$$

has a unique solution $c \in \mathbb{R}^n$ (or $\mathbb{C}^n$).

Remark. $\{v_1 \dots v_n\}$ is basis if and only if $\det V \neq 0$.

$\Longleftrightarrow$ V is invertible.

Scalar product.

Let V be a vector space.

Definition A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is called a **scalar product** (or inner product) on V if:

(1) (positivity):

(i) For any $x \in V$, $\langle x, x \rangle \geq 0$

(ii) $\langle x, x \rangle = 0$ if and only if $x = 0$.

(2) (Skew-symmetry) For any $x, y \in V$

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$ ← complex conjugate

(3) (linearity) For any $x, y, z \in V$, $a, b \in \mathbb{C}$:

$$\langle ax + by, z \rangle = a \cdot \langle x, z \rangle + b \langle y, z \rangle.$$

Example: If $V = \mathbb{C}^n$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, $\langle x, y \rangle = x_1 \overline{y_1} + x_2 \overline{y_2} + \cdots + x_n \overline{y_n}$