L3

Scalar Product. Orthonormal Basis. Approximations

Assume $(V, \langle \cdot, \cdot \rangle)$ is a vector space with scalar product.

<u>Theorem</u> [Cauchy-Schwartz Inequality]. Let $x, y \in V$. Then:

$$|\langle x,y\rangle| \le \sqrt{\langle x,x\rangle} \cdot \sqrt{\langle y,y\rangle}.$$

<u>Why?</u>

<u>Assume.</u> $\langle x,y\rangle$ is <u>real</u>. (Not too complicated to extend to $\mathbb{C}$)

Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, $\varphi(t) = \langle x+ty, x+ty\rangle$.

We know: $\varphi(t) \ge 0$, for every $t \in \mathbb{R}$. (by positivity of the inner product).

$$\varphi(t) = \langle x+ty, x+ty\rangle \overset{\text{by linearity}}{=} \langle x, x+ty\rangle + t\langle y, x+ty\rangle =$$

$$\overset{\text{by linearity}}{=} \langle x,x\rangle + t\langle x,y\rangle + t\langle y,x\rangle + t^2\langle y,y\rangle =$$

$$\overset{\text{by (skew)symmetry.}}{=} \langle x,x\rangle + 2t\langle x,y\rangle + t^2\langle y,y\rangle.$$

Let $a = \langle y,y\rangle$, $b = 2\langle x,y\rangle$, $c = \langle x,x\rangle$.

Then: $\varphi(t) = at^2 + bt + c \ge 0$, $\forall t \in \mathbb{R}$ (for every real t).

<u>Case 1</u>: $a = 0$: $\langle y,y\rangle = 0 \Rightarrow y = 0$.

$\Rightarrow \langle x,y\rangle = 0.$

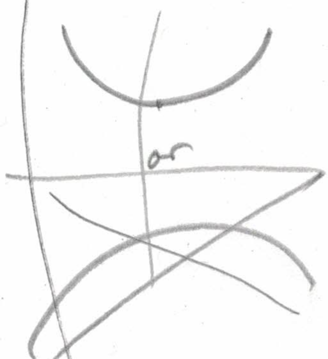
(why: $\underset{2\cdot\langle x,0\rangle}{\langle 2x,y\rangle} = \langle x, 2y\rangle = \langle x,0\rangle$)

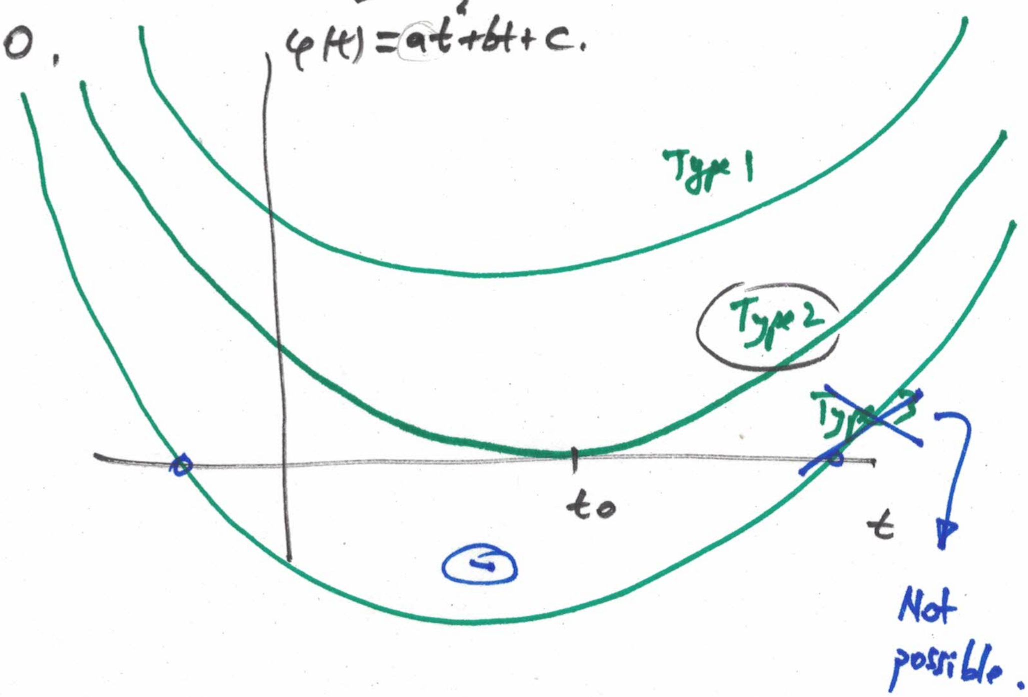
$\Rightarrow \quad 0 \le \sqrt{\langle x,x\rangle}\cdot 0 \Rightarrow \boxed{\text{ok}}$

Case 2: $a \neq 0 \Leftrightarrow y \neq 0$.

$[\, at^2 + bt + c \geq 0, \ \forall t \in \mathbb{R}$ — quadratic function.

But: $a = \langle y, y \rangle > 0$, $\quad \varphi(t) = at^2 + bt + c$.





Parabola is either of $\underbrace{\text{type 1 or type 2}}$.

$\Leftrightarrow \underline{\Delta = b^2 - 4ac} \leq 0$.

$(2\langle x, y \rangle)^2 - 4\langle y, y \rangle \cdot \langle x, x \rangle \leq 0$.

$(\langle x, y \rangle)^2 \leq \langle x, x \rangle \cdot \langle y, y \rangle$.

$\Updownarrow$

$$\boxed{|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \cdot \sqrt{\langle y, y \rangle}\,.}$$

CS inequality.

Equality:

$\Delta = 0 \Leftrightarrow at^2 + bt + c = 0$, for some t_0.

(Type 2)

$\langle x + t_0 y, x + t_0 y \rangle = 0 \Leftrightarrow \boxed{x + t_0 y = 0}$

Other examples of scalar products:

(1) $V = \mathbb{R}^n$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$\langle x, y \rangle_1 = x_1 y_1 + 2 \cdot x_2 y_2 + 3 \cdot x_3 y_3 + \dots + n \cdot x_n y_n .$$

(2) For any $a_1, a_2, \dots, a_n > 0$ (real):

$V = \mathbb{R}^n$,

$$\langle x, y \rangle_2 = a_1 x_1 y_1 + a_2 x_2 y_2 + \dots + a_n x_n y_n .$$

(3) If $A = A^T \in \mathbb{R}^{n \times n}$, $V = \mathbb{R}^n$

consider:

$$\langle x, y \rangle_A = x^T \cdot A \cdot y = (x_1, \dots x_n) \cdot \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & & \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i y_j \qquad (*)$$

<u>Fact</u>: The matrix $A = A^T$ defines a scalar product via $\langle \cdot, \cdot \rangle_A$ as in $(*)$ <u>if and only if</u>. A is a <u>positive definite matrix</u>.

$\Updownarrow$

Each eigenvalue of A is positive.

$\Updownarrow$ $\boxed{\text{CHOLESKY}}$

there is $W \in \mathbb{R}^{n \times n}$ s.t. $A = W \cdot W^T$ (<u>Cholesky factorization</u>).

NORM: Let V be a vector space.

Definition. A function $\|\cdot\| : V \longrightarrow \mathbb{R}$ is called a norm on V if it satisfies:

(1) Positivity: (i) $\forall x \in V, \quad \|x\| \geq 0$.

(ii) $\|x\| = 0$ if and only if $x = 0$.

(2) Homogeneity: $\forall x \in V, \ \forall a \in \mathbb{C}$:

$$\|a \cdot x\| = |a| \cdot \|x\|$$

(3) Triangle inequality: $\forall \ x, y \in V$

$$\|x + y\| \leq \|x\| + \|y\|.$$

Remark: If $(V, \langle \cdot, \cdot \rangle)$ is a vector space with scalar product then

$$x \longmapsto \sqrt{\langle x, x \rangle}$$

defines a norm, known as the associated norm (induced norm).

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Example: $V = \mathbb{R}^n$, $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$.

Then $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ (Euclidean norm)

(distance)

Notation: $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

If $V = \mathbb{C}^n$:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n \dashrightarrow \|x\| = \|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$$

Example: $V = \mathbb{C}^n$, $1 \leq p$

Set:

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p \right)^{1/p}$$

Example: $p = \infty$.

$$\|x\|_\infty = \max_{1 \leq k \leq n} |x_k|$$

Fact: For each $1 \leq p \leq \infty$, $\|\cdot\|_p$ defines a norm.

Example. For $1 \leq p < \infty$.

sequence of complex numbers indexed by $\mathbb{Z}$

$$\ell^p(\mathbb{Z}) = \left\{ x = (x_n)_{n \in \mathbb{Z}} : \sum_{n=-\infty}^{\infty} |x_n|^p < \infty \right\}.$$

Norm:

$$\|x\|_p = \left(\sum_{n=-\infty}^{\infty} |x_n|^p \right)^{1/p}$$

: **Fact:** $\|\cdot\|_p$ defines a norm on $\ell^p(\mathbb{Z})$.

Example $p = \infty$:

$$\ell^\infty(\mathbb{Z}) = \{ x = (x_n)_{n \in \mathbb{Z}} : \underbrace{\exists M > 0 : |x_n| \leq M, \text{ for every } n}_{(x_n)_n \text{ is bounded}} \}$$

$$\|x\|_\infty = \sup_n |x_n| = \text{supremum of } \{ |x_n|, n \in \mathbb{Z} \}.$$

Defines a norm

ONB : Orthonormal Basis.

Let: $V = \mathbb{C}^n$ and $\langle \cdot , \cdot \rangle$ is a scalar product on V.

Definition A set $\{e_1, e_2, \dots, e_n\} \subset V$ is called an **orthonormal basis (ONB)** if:

BASIS

(1) $\{e_1, e_2 \dots e_n\}$ is basis for V

ORTHONORMALITY.

(2) $\langle e_i, e_j \rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$, for every $1 \le i, j \le n$.

Remark: If V is finite dimensional (which is the case here, $V = \mathbb{C}^n$)

(2) $\Rightarrow$ (1) : [ORTHONORMAL & there are n vectors] $\Rightarrow$ BASIS.

What is an ONB good for?

Let $(V, \langle \cdot ; \cdot \rangle)$ be a vector space with scalar product.

Assume $\{e_1, e_2 \dots e_n\}$ is ONB in V.

Fix $1 \le d \le n$. Let $F = \text{span}\{e_1, e_2 \dots e_d\} = \left\{ \sum_{k=1}^{d} c_k e_k = c_1 e_1 + \dots + c_d e_d \right.$

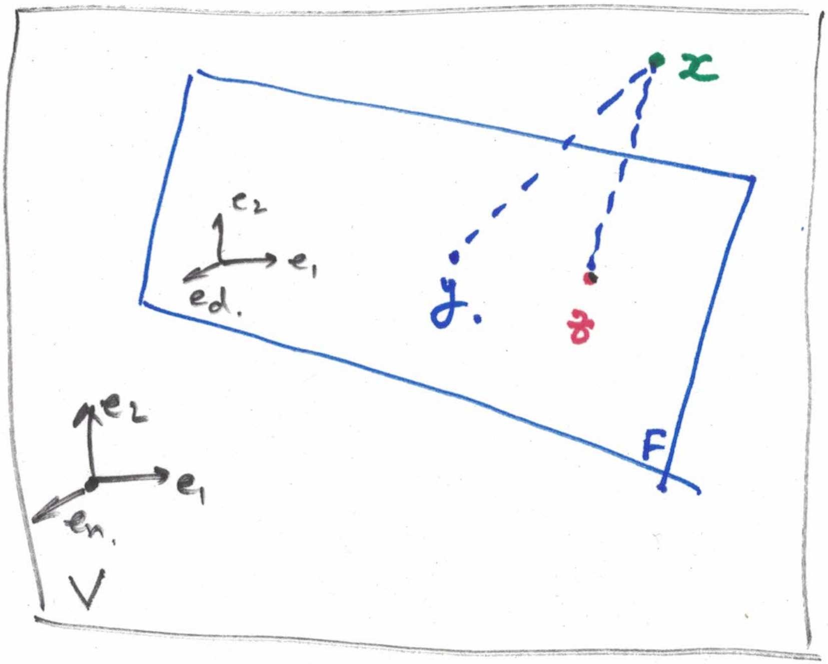
$\left. c_1 \dots c_d \in \mathbb{C} \right\}$.

Note: · F is a vector space, subset of V.

· $\dim F = d$.

Problem: Given $x \in V$, find $y \in F$ that minimizes:

$$\min_{y \in F} \|y - x\|$$



Want: Find a (the) point $z \in F$ such that:

$$\|z - x\| \leq \|y - x\|, \text{ for every } y \in F$$

Theorem. Assume the setup introduced before: $(V, \langle \cdot ; \cdot \rangle)$ is a vector with scalar product; $\{e_1 \dots e_n\}$ ONB in V; $1 \leq d \leq n$; $F = \mathrm{Span}\{e_1 \dots e_d\}$. Let $x \in V$.

Then the vector $z = \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2 + \dots + \langle x, e_d \rangle e_d$

satisfies:

$$\|z - x\| \leq \|y - x\|, \text{ for every } y \in F.$$

Notation: The map $x \longmapsto z$ is called the orthogonal projection of x onto F. Notation: $P_F : x \longmapsto P_F(x) = \langle x, e_1 \rangle e_1 + \dots + \langle x, e_d \rangle e_d$

Remark: If $d = n$. Then $F = V$ and $z = x$:

$$x = \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2 + \dots + \langle x, e_n \rangle e_n. \Bigg] \to \text{ONB decomposition of } x.$$

Why (Theorem)

$$c_1 e_1 + \cdots + c_d e_d \quad \text{for any } c_1 \dots c_d \in \mathbb{C}$$

Need: $\|z - x\|^2 \leq \|y - x\|^2$

$$\langle x, e_1\rangle e_1 + \cdots + \langle x, e_d\rangle e_d.$$

Check in a simplified case: $d = 2$, $n = 3$. and $\langle x, y\rangle = x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3$

$$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

$$x = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}, \dashrightarrow z = x_1 \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\0\end{pmatrix}.$$

$$F = \operatorname{span}\{e_1, e_2\} = \left\{ c_1 e_1 + c_2 e_2 = \begin{pmatrix}c_1\\c_2\\0\end{pmatrix},\ c_1, c_2 \in \mathbb{C} \right\}.$$

$$\underbrace{\left\| \begin{pmatrix}x_1\\x_2\\0\end{pmatrix} - \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \right\|^2}_{|x_3|^2} \leq \left\| \begin{pmatrix}c_1\\c_2\\0\end{pmatrix} - \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \right\|^2.$$

$$|x_3|^2 \leq \underbrace{|c_1 - x_1|^2 + |c_2 - x_2|^2}_{\geq 0} + |x_3|^2$$

always true.

$\geq 0. \quad \Rightarrow$ z is unique!