Proof of Theorem.

Need to show

$$\|z-x\|^2 \le \|y-x\|^2 \qquad (1)$$

as:

$$\|\underbrace{\langle x,e_1\rangle e_1 + \dots + \langle x,e_d\rangle e_d}_{z} - x\|^2 \le \|\underbrace{c_1e_1 + \dots + c_de_d}_{y} - x\|^2$$

for every $c_1, c_2, \dots c_d \in \mathbb{C}$.

Step 1. Expand both sides of (1):

$$\|z-x\|^2 = \langle z-x, z-x\rangle = \langle z,z\rangle - \langle z,x\rangle - \langle x,z\rangle + \langle x,x\rangle \qquad (2)$$

$$\|y-x\|^2 = \quad \dots \quad = \langle y,y\rangle - \langle y,x\rangle - \langle x,y\rangle + \langle x,x\rangle. \qquad (3)$$

But:

$$\langle z,z\rangle = \langle \langle x,e_1\rangle e_1 + \dots + \langle x,e_d\rangle e_d, \langle x,e_1\rangle e_1 + \dots + \langle x,e_d\rangle e_d\rangle =$$

$$= \langle x,e_1\rangle \cdot \overline{\langle x,e_1\rangle} \cdot \underbrace{\langle e_1,e_1\rangle}_{1} + \langle x,e_1\rangle \cdot \overline{\langle x,e_2\rangle} \overbrace{\langle e_1,e_2\rangle}^{=0} + \dots + \langle x,e_1\rangle \cdot \overline{\langle x,e_d\rangle} \cdot \underbrace{\langle e_1,e_d\rangle}_{=0}$$

$$+ \dots + \langle x,e_d\rangle \overline{\langle x,e_1\rangle} \cdot \underbrace{\langle e_d,e_1\rangle}_{=0} + \dots + \langle x,e_d\rangle \cdot \overline{\langle x,e_d\rangle} \underbrace{\langle e_d,e_d\rangle}_{1} \overset{\text{By ONB}}{=\!=}$$

$$= |\langle x,e_1\rangle|^2 \cdot 1 + \dots + |\langle x,e_d\rangle|^2 \cdot 1 = \sum_{k=1}^{d} |\langle x,e_k\rangle|^2$$

Similarly:

$$\langle y,y\rangle = \langle c_1e_1 + \dots + c_de_d, c_1e_1 + \dots + c_de_d\rangle \overset{\text{by ONB}}{=\!=}$$

$$= |c_1|^2 + \dots + |c_d|^2 = \sum_{k=1}^{d} |c_k|^2$$

and

$$\langle z,x\rangle = \langle \langle x,e_1\rangle e_1 + \dots + \langle x,e_d\rangle e_d, x\rangle = \langle x,e_1\rangle \cdot \langle e_1,x\rangle + \dots + \langle x,e_d\rangle \langle e_d,x\rangle =$$

$$= |\langle x,e_1\rangle|^2 + \dots + |\langle x,e_d\rangle|^2 = \overline{\langle x,z\rangle} = \langle x,z\rangle.$$

↑ because $|\langle x,e_1\rangle|^2 + \dots + |\langle x,e_d\rangle|^2$ is real.

and

$$\langle y,x\rangle = \langle c_1e_1 + \dots + c_d e_d, x\rangle = c_1\langle e_1,x\rangle + \dots + c_d\langle e_d,x\rangle.$$

Step 2. Substitute these expressions in (2) and (3):

$$\|y-x\|^2 = |c_1|^2 + \dots + |c_d|^2 - c_1\langle e_1,x\rangle - \dots - c_d\langle e_d,x\rangle -$$

$$-\overline{c_1}\langle x,e_1\rangle - \dots - \overline{c_d}\cdot\langle x,e_d\rangle + \|x\|^2$$

$$\|z-x\|^2 = |\langle x,e_1\rangle|^2 + \dots + |\langle x,e_d\rangle|^2 - |\langle x,e_1\rangle|^2 - \dots - |\langle x,e_d\rangle|^2 -$$

$$-|\langle x,e_1\rangle|^2 - \dots - |\langle x,e_d\rangle|^2 + \|x\|^2 = \|x\|^2 - |\langle x,e_1\rangle|^2 - \dots - |\langle x,e_d\rangle|^2$$

Step 3. Simplify:

$$\|y-x\|^2 - \|z-x\|^2 = |c_1|^2 + \dots + |c_d|^2 - c_1\cdot\overline{\langle x,e_1\rangle} - \dots - c_d\overline{\langle x,e_d\rangle} -$$

$$-\overline{c_1}\langle x,e_1\rangle - \dots - \overline{c_d}\cdot\langle x,e_d\rangle + |\langle x,e_1\rangle|^2 + \dots + |\langle x,e_d\rangle|^2 =$$

$$= |c_1 - \langle x,e_1\rangle|^2 + |c_2 - \langle x,e_2\rangle|^2 + \dots + |c_d - \langle x,e_d\rangle|^2$$

because $|a-b|^2 = |a|^2 - a\overline{b} - \overline{a}\cdot b + |b|^2$ when $a,b\in\mathbb{C}$.

Thus, we obtained:

$$\|y-x\|^2 - \|z-x\|^2 = \sum_{k=1}^{d} \underbrace{|c_k - \langle x,e_k\rangle|^2}_{\geq 0} \geq 0.$$

This proves: $\|y-x\| \geq \|z-x\|$. Furthermore, $\|y-x\| = \|z-x\|$ if and only if $c_1 = \langle x,e_1\rangle$ and … $c_d = \langle x,e_d\rangle$