L8

The Fourier Transform

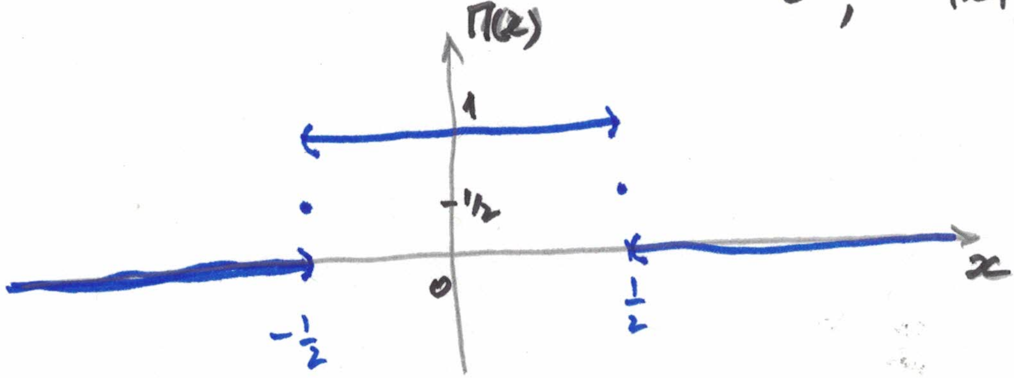
Def. The (continuous) Fourier Transform of a function $f: \mathbb{R} \to \mathbb{C}$ is the function $F: \mathbb{R} \to \mathbb{C}$ defined by:

$$F(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} f(x)\, dx.$$

Notation: $\hat{f}: \mathbb{R} \to \mathbb{C}$, $\hat{f} = F$.

Example. The Box function.

$$\Pi: \mathbb{R} \to \mathbb{R}, \qquad \Pi(x) = \begin{cases} 1, & -\frac{1}{2} < x < \frac{1}{2} \\ \frac{1}{2}, & x = \pm 1/2 \\ 0, & |x| > \frac{1}{2} \end{cases}$$



Notation: $\Pi = 1_{(-\frac{1}{2},\frac{1}{2})}$ $\left(+\frac{1}{2} 1_{\{-\frac{1}{2},\frac{1}{2}\}}\right)$.

Compute its Fourier transform:

$$\hat{\Pi}(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} \cdot \Pi(x)\, dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i x s}\, dx$$

$s=0$:

$$\hat{\Pi}(0) = \int_{-1/2}^{1/2} e^{0}\, dx = 1.$$

$s \neq 0$:

$$\hat{\Pi}(s) = \int_{-1/2}^{1/2} e^{-2\pi i x s}\, dx = \frac{1}{-2\pi i s}\left(e^{-2\pi i x s}\Big|_{-1/2}^{1/2} \right) =$$

$$= \frac{e^{-\pi i s} - e^{\pi i s}}{-2\pi i s} = \frac{\cancel{\cos}(\pi s) - i\sin(\pi s) - (\cancel{\cos}(\pi s) + i\sin(\pi s))}{-2\pi i s}$$

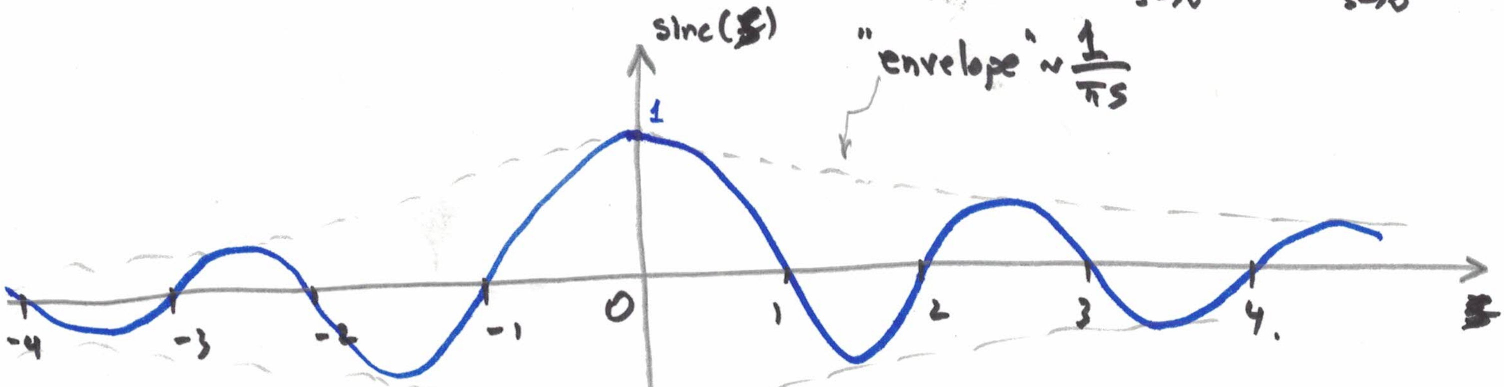
$$= \frac{-2i\ \sin(\pi s)}{-2i\ \pi s} = \frac{\sin(\pi s)}{\pi s}$$

The "sinc" function: $\quad \mathrm{Sinc} : \mathbb{R} \to \mathbb{R}$,

$$\mathrm{Sinc}(s) = \begin{cases} \dfrac{\sin(\pi s)}{\pi s}, & s \neq 0 \\ 1, & s = 0 \end{cases}$$

We obtained: $\quad \hat{\Pi} = \mathrm{sinc}$.

$$\lim_{s\to 0} \frac{\sin(\pi s)}{\pi s} = \lim_{s\to 0} \cos(\pi s) \ldots$$



$$(s>0);\quad \left|\frac{\sin(\pi s)}{\pi s}\right| \le \frac{1}{\pi s}$$

<u>Remark</u>: The sinc function is "very smooth":

sinc is ∞-many times differentiable, $\in C^\infty$

for $s \neq 0$:

$$(\mathrm{sinc}(s))' = \frac{d}{ds}\left(\frac{\sin(\pi s)}{\pi s}\right) = \frac{\pi \cos(\pi s)\, \pi s - \pi \sin(\pi s)}{\pi^2 s^2} =$$

$$= \frac{\pi s \cos(\pi s) - \sin(\pi s)}{\pi s^2}$$

$$\lim_{s \to 0} \frac{\pi s \cos(\pi s) - \sin(\pi s)}{\pi s^2} = \lim_{s \to 0} \frac{\cancel{\pi \cos(\pi s)} - \pi^2 s \sin(\pi s) - \cancel{\pi \cos(\pi s)}}{2\pi s} =$$

$$= \lim_{s \to 0} \left(-\frac{\pi}{2} \cdot \sin(\pi s)\right) = \underline{\underline{0}}.$$

$$\lim_{s \to 0} \frac{\mathrm{sinc}(s) - \mathrm{sinc}(0)}{s - 0} = \lim_{s \to 0} \frac{\frac{\sin(\pi s)}{\pi s} - 1}{s} = \lim_{s \to 0} \frac{\sin(\pi s) - \pi s}{\pi s^2} =$$

$$= \lim_{s \to 0} \frac{\cancel{\pi} \cos(\pi s) - \cancel{1}}{2\cancel{\pi} s} = \lim_{s \to 0} \left(\frac{-\pi \sin(\pi s)}{2}\right) = \underline{\underline{0}}$$

$\Rightarrow$ sinc is at least C^1

... ...$\to$ can check that sinc is (at least) C^∞.

In fact sinc is an <u>analytic</u> function

$\hookrightarrow$ Its Taylor series converges to the function itself.

analytic on $\mathbb{C}$

sinc is a holomorphic function. (in fact an entire function)

① What functions can be Fourier transformed?

$$f \longrightarrow F(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} f(x)\, dx.$$

$$|F(s)| = \left| \int_{-\infty}^{\infty} e^{-2\pi i x s} f(x)\, dx \right| \leq \int_{-\infty}^{\infty} \left| e^{-2\pi i x s} \cdot f(x) \right| dx =$$

by triangle inequality

$$\left| e^{-2\pi i x s} \cdot f(x) \right| = \underbrace{\left| e^{-2\pi i x s} \right|}_{1} \cdot |f(x)| = |f(x)|$$

$$= \int_{-\infty}^{\infty} |f(x)|\, dx$$

<u>Recap</u>: For every $s \in \mathbb{R}$, $|F(s)| \leq \int_{-\infty}^{\infty} |f(x)|\, dx.$

Any function $f \in L^1(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; \int_{-\infty}^{\infty} |f(x)|\, dx < \infty \right\}$

admits a Fourier transform, F and:

$$\underbrace{\max_{s \in \mathbb{R}} |F(s)|}_{} \leq \int_{-\infty}^{\infty} |f(x)|\, dx.$$

sup

$$\|F\|_\infty \leq \|f\|_1$$

$$\|F\|_\infty = \sup_{s \in \mathbb{R}} |F(s)| \quad , \quad \|f\|_1 = \int_{-\infty}^{\infty} |f(x)|\, dx.$$

...e Obtained: If $\mathcal{F}$ denotes the Fourier transform operator,

$$f \longmapsto \mathcal{F}(f) = F$$

$$\mathcal{F} : L^1(\mathbb{R}) \longrightarrow L^\infty(\mathbb{R})$$

$$\|\mathcal{F}(f)\|_\infty \leq \|f\|_1 .$$

where:

$$L^1(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; \|f\|_1 := \int_{-\infty}^{\infty} |f(x)|\,dx < \infty . \right\}$$

$$L^\infty(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; \|f\|_\infty := \operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x)| < \infty . \right\}$$

"essential supremum"

Facts:

① If $f \in L^1(\mathbb{R})$, then $F : \mathbb{R} \to \mathbb{C}$ is always continuous. In fact it is uniformly continuous.

$$\left[\; \forall \varepsilon > 0 \;\; \exists \delta_\varepsilon > 0 \;\; \forall x, y \in \mathbb{R}, \;\; |x-y| < \delta_\varepsilon \Rightarrow |F(x) - F(y)| < \varepsilon . \right]$$

② [Riemann-Lebesgue Lemma]. If $f \in L^1(\mathbb{R})$, then

$$\lim_{s \to \infty} F(s) = \lim_{s \to -\infty} F(s) = 0 .$$

Remark:

Comparison between Fourier transform and Laplace transform.

$$f \xrightarrow{\mathcal{F}} \hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} f(x)\,dx$$

$$f \xrightarrow{\mathcal{L}} F(s) = \int_{0}^{\infty} e^{-xs} f(x)\,dx.$$

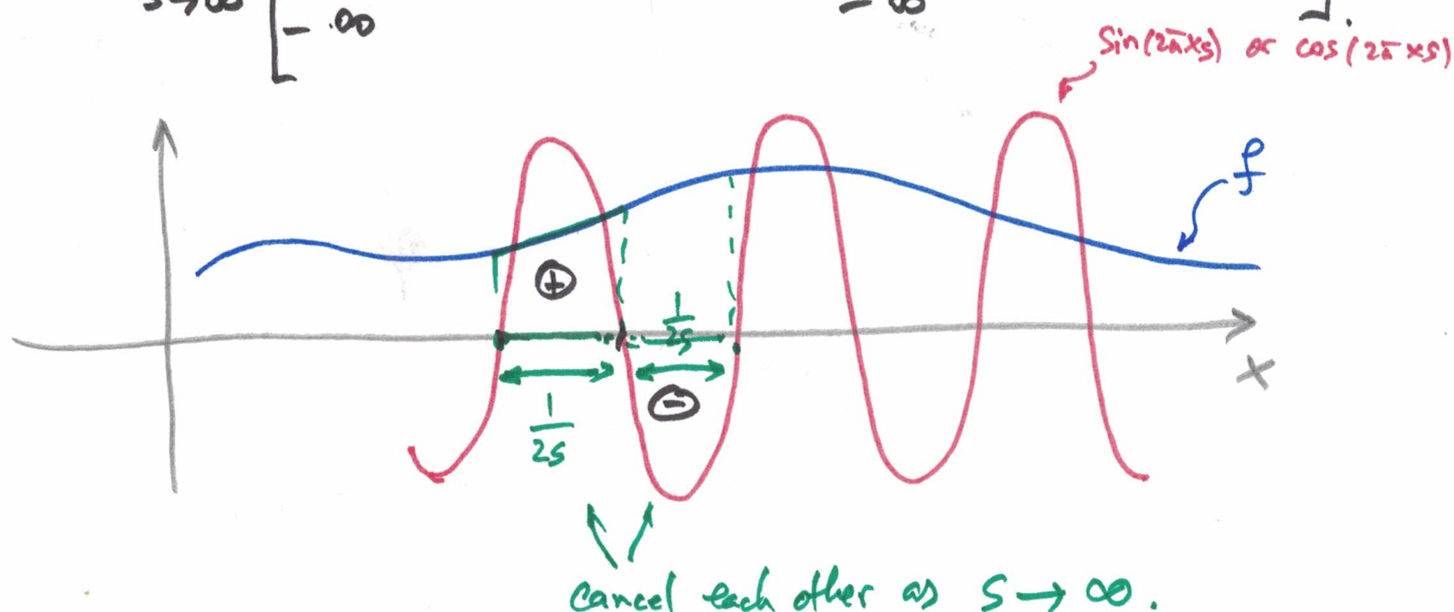
Laplace transform operator

The Laplace transform of f.

Remark about Riemann-Lebesgue lemma:

$$0 \stackrel{?}{=} \lim_{s\to\infty} F(s) = \lim_{s\to\infty} \int_{-\infty}^{\infty} e^{-2\pi i x s} f(x)\,dx =$$

$$= \lim_{s\to\infty} \left[\int_{-\infty}^{\infty} \cos(2\pi x s) f(x)\,dx - i \int_{-\infty}^{\infty} \sin(2\pi x s) f(x)\,dx \right]$$



<u>Theorem</u> [Plancherel–Parseval]. Assume

$$f : \mathbb{R} \to \mathbb{C}, \text{ s.t. } \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \qquad \left(\text{i.e. } f \in L^2(\mathbb{R})\right).$$

Then the Fourier transform of f, F, <u>exists</u> and:

$$\int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 \, dx.$$

F ~~existence~~ is defined in an L^2-sense (mean square convergence):

$$\lim_{R \to \infty} \int_{-\infty}^{\infty} \left| F(s) - \int_{-R}^{R} e^{-2\pi i x s} f(x) \, dx \right|^2 ds = 0.$$

<u>Remark</u>: Compare: Fourier series: $g : [0,1] \xrightarrow{\text{s.t. } \int_0^1 |g(x)|^2 dx < \infty} \mathbb{C} \longrightarrow c_n = \int_0^1 e^{-2\pi i n x} g(x) \, dx$

$$\longrightarrow \lim_{N \to \infty} \int_0^1 \left| g(x) - \sum_{k=-N}^{N} c_k e^{2\pi i k x} \right|^2 dx = 0$$

$$\int_0^1 |g(x)|^2 dx = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

Using $\mathcal{F} : f \longmapsto \mathcal{F}(f) = F$ (the Fourier transform).

Plancherel Theorem: $\mathcal{F} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$

and $\|\mathcal{F}(f)\|_2 = \|f\|_2$ (is an isometry).

in fact it is $\boxed{\text{UNITARY}}$

where:

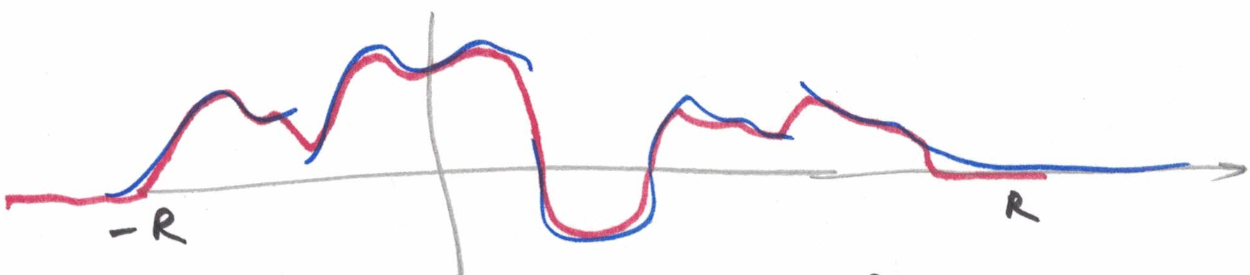
$$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; \underbrace{\|f\|_2^2}_{L^2\text{-norm squared.}} := \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty . \right\}.$$

<u>Why?</u>

$f \in L^2(\mathbb{R})$ ---→ We can approximate arbitrary well by g

1) $g : \mathbb{R} \to \mathbb{C},\; g \in C^1$

2) $g(x) = 0,\; \forall\, x \leq -R$ or $x \geq R$.



$$\to \hat{g}(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} g(x)\,dx = \int_{-R}^{R} e^{-2\pi i x s} g(x)\,dx$$

at $s = \frac{n}{2R}$: $\underbrace{\hat{g}\left(\frac{n}{2R}\right)}_{n^{th}\text{ Fourier coeff of } g.} = \int_{-R}^{R} e^{-\frac{2\pi i n x}{2R}} g(x)\,dx$

$$\lim_{R\to\infty}\left[\lim_{N\to\infty} \int_{-\infty}^{\infty} \Big| g(x) - \underbrace{\sum_{n=-N}^{N} \hat{g}\left(\tfrac{n}{2R}\right) e^{\frac{2\pi i n x}{2R}}}_{\longrightarrow\ \sim \int_{-\infty}^{\infty} e^{2\pi i s x} \hat{g}(s)\,ds} \Big|^2 dx = 0 .\right]$$

$$\to \left\| g(x) - \int_{-\infty}^{\infty} e^{2\pi i x s} \hat{g}(s)\,ds \right\| = 0$$

$$\frac{C_0}{\pi}\log(T) = C_0 \cdot \int_1^T \frac{ds}{\pi s} \le \int_1^T \left|\frac{\sin(\pi s)}{\pi s}\right| ds. \le \frac{1}{\pi}\int_1^T \frac{ds}{s} = \frac{1}{\pi}\log(T)$$

, For all $T > 1$.

for some $0 < C_0 < 1$.

$$\frac{C_0}{\pi}\log(T) \le \int_1^T |\operatorname{sinc}(s)|\, ds \le \frac{1}{\pi}\log(T).$$

$$\downarrow$$

$$\int_1^\infty |\operatorname{sinc}(s)|\, ds = +\infty.$$

$\rightarrow$ $\operatorname{sinc} \notin L^1(\mathbb{R})$.