Fourier Transform: Rules and Examples

Recap: Given $f: \mathbb{R} \to \mathbb{C}$,

Its Fourier transform, $F: \mathbb{R} \to \mathbb{C}$,

$$F(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} f(x)\, dx.$$

last time:

- $f \in L^1(\mathbb{R}) \longrightarrow F \in L^\infty(\mathbb{R})$,

$$|F(s)| \leq \int_{-\infty}^{\infty} |f(x)|\, dx =: \|f\|_1 .$$

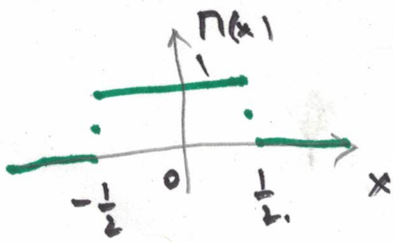
$$\rightarrow \|F\|_\infty := \max_{s \in \mathbb{R}} |F(s)| \leq \|f\|_1 .$$

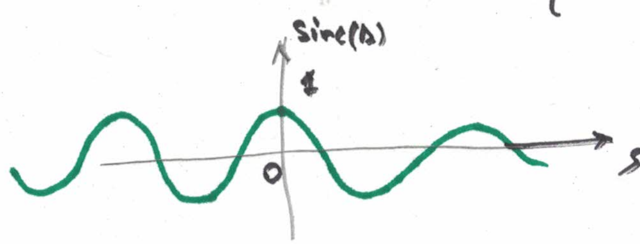
- $f \in L^2(\mathbb{R}) \longrightarrow F \in L^2(\mathbb{R})$.

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds .$$

$$\|F\|_2 = \|f\|_2 .$$

- $f(x) = \Pi(x) \longrightarrow F(s) = \mathrm{sinc}(s) = \begin{cases} \dfrac{\sin(\pi s)}{\pi s}, & s \neq 0 \\ 1, & s = 0 \end{cases}$





Rules: Linearity.

Function	Fourier Transform.
f	$\longmapsto F$
g	$\longrightarrow G$

Then: $\alpha, \beta \in \mathbb{C}$

$$\alpha f + \beta g \longrightarrow \alpha F + \beta \cdot G.$$

Why:

$$\mathcal{F}(\alpha f + \beta g)(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} (\alpha f(x) + \beta g(x))\,dx = \alpha \int_{-\infty}^{\infty} e^{-2\pi i x s} f(x)\,dx + \beta \int_{-\infty}^{\infty} e^{-2\pi i x s} g(x)\,dx = \alpha \cdot F(s) + \beta \cdot G(s).$$

Remark: NOTE: $\boxed{\alpha, \beta \text{ must be scalars. } \boxed{\text{NOT Functions!!!}}}$

If $f \longrightarrow F$

$g \longrightarrow G$

$$\boxed{f \cdot g \overset{\text{NO}}{\not\longrightarrow} F \cdot G}$$

Rule: Translation

Function	Fourier Transform
f	$\longrightarrow F$

Let $\tau \in \mathbb{R}$

$$g(x) = f(x - \tau) \longrightarrow G(s) = e^{-2\pi i s \cdot \tau} \cdot F(s).$$

Why: $$G(s)=\int_{-\infty}^{\infty} e^{-2\pi i s x}\, g(x)\,dx = \int_{-\infty}^{\infty} e^{-2\pi i s x}\, f(x-\tau)\,dx = \int_{-\infty}^{\infty} e^{-2\pi i s (y+\tau)}\, f(y)\,dy =$$

$$y = x-\tau$$
$$dy = dx$$

$$=? \quad e^{-2\pi i s(y+\tau)} = e^{-2\pi i s y - 2\pi i s \tau} = e^{-2\pi i s y}\cdot e^{-2\pi i s \tau}$$

$$= \int_{-\infty}^{\infty} \underbrace{e^{-2\pi i s \tau}}_{\text{independent of } y.} \cdot e^{-2\pi i s y} f(y)\,dy = e^{-2\pi i s \tau}\cdot \int_{-\infty}^{\infty} e^{-2\pi i s y} f(y)\,dy =$$

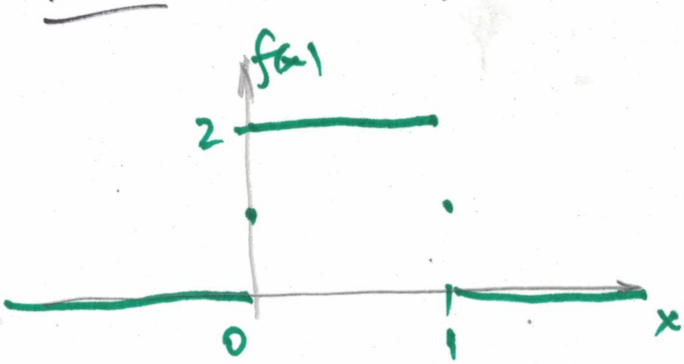
$$= e^{-2\pi i s \tau}\cdot F(s)$$

Example: let $$f(x) = \begin{cases} 2, & 0<x<1. \\ 0, & x<0 \text{ or } x>1. \end{cases}$$

Remark: If we do not specifically define f at some point x_0, assume Dirichlet regularization: $f(x_0) = \frac{1}{2}\left(f(x_0-0)+f(x_0+0)\right)$

In the case f above: $f(0)=1$, $f(1)=1$.

Problem: Find F, the Fourier transform of f.



$$f(x) = 2\cdot \Pi\left(x-\tfrac{1}{2}\right).$$

Apply linearity & translation rule: $\tau = \frac{1}{2}$

$$F(s) = 2\, e^{-\pi i s}\cdot \operatorname{sinc}(s).$$

Rule : Modulation.

Function $\longrightarrow$ Fourier Transform

$$f \longrightarrow F$$

let $\omega \in \mathbb{R}$

$$\longrightarrow g(x) = e^{2\pi i \omega x} \cdot f(x) \longrightarrow G(s) = F(s-\omega)$$

Why:

$$G(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} g(x) dx = \int_{-\infty}^{\infty} e^{-2\pi i x s} e^{+2\pi i \omega x} f(x) dx =$$

$$= \int_{-\infty}^{\infty} e^{-2\pi i x s + 2\pi i x \omega} \cdot f(x) dx = \int_{-\infty}^{\infty} e^{-2\pi i x (s-\omega)} f(x) dx = F(s-\omega). \blacksquare$$

Example.

$$f(x) = \begin{cases} \cos(\pi x), & -\frac{1}{2} < x < \frac{1}{2} \\ 0, & |x| > \frac{1}{2} \end{cases}$$

Problem: Find F, the Fourier transform of f.

$$f(x) = \cos(\pi x) \cdot \Pi(x) = \frac{1}{2}\left(e^{i\pi x} + e^{-i\pi x}\right) \cdot \Pi(x) =$$

$$= \frac{1}{2} e^{i\pi x} \cdot \Pi(x) + \frac{1}{2} e^{-i\pi x} \cdot \Pi(x).$$

by linearity and modulation rule

$$\longrightarrow F(s) = \underbrace{\frac{1}{2} \operatorname{sinc}\left(s - \frac{1}{2}\right)}_{\text{with } \omega = \frac{1}{2}} + \underbrace{\frac{1}{2} \operatorname{sinc}\left(s + \frac{1}{2}\right)}_{\text{with } \omega = -\frac{1}{2}}.$$

Rule : Scaling.

	Function		Fourier Transform
If.	f	$\longrightarrow$	F

and $a \in \mathbb{R}$, $a \neq 0$

Then: $g(x) = f(a \cdot x) \longrightarrow G(s) = \frac{1}{|a|} \cdot F\left(\frac{s}{a}\right)$

Why:

$$G(s) = \int_{-\infty}^{\infty} e^{-2\pi i s x} g(x)\, dx = \int_{-\infty}^{\infty} e^{-2\pi i s x} f(a \cdot x)\, dx =$$

let $y = a \cdot x$

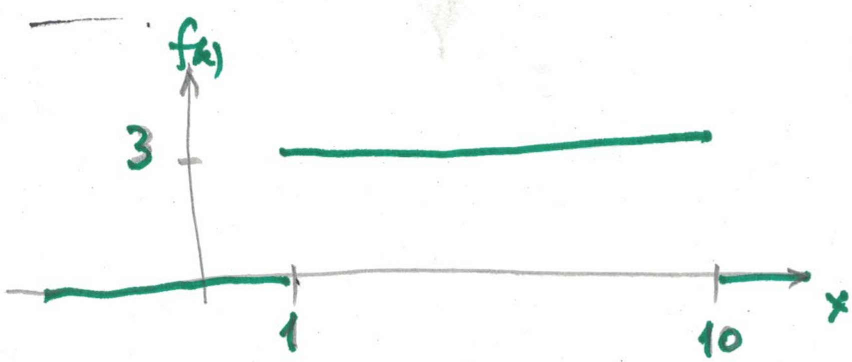
$dy = |a|\, dx$

$$= \int_{-\infty}^{\infty} e^{-2\pi i s \frac{y}{a}} f(y) \frac{1}{|a|}\, dy = \frac{1}{|a|} \int_{-\infty}^{\infty} e^{-2\pi i \frac{s}{a} y} f(y)\, dy = \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

Example.

$$f(x) = \begin{cases} 3, & 1 < x < 10. \\ 0, & x < 1 \text{ or } x > 10 \end{cases}$$

$F = ?$



$f(x) = 3 \cdot \Pi(\alpha x + \beta)$.

for some $\alpha, \beta \in \mathbb{R}$.

$x = 1 \longrightarrow \alpha x + \beta = \alpha + \beta = -\frac{1}{2}$

$x = 10 \longrightarrow \alpha x + \beta = 10\alpha + \beta = \frac{1}{2}$

$$\alpha + \beta = -\frac{1}{2}$$
$$10\alpha + \beta = \frac{1}{2}$$

$$9\alpha = 1 \longrightarrow \alpha = \frac{1}{9} \longrightarrow \beta = -\frac{1}{2} - \frac{1}{9} = -\frac{11}{18}$$

$$f(x) = 3 \cdot \Pi\left(\frac{1}{9}x - \frac{11}{18}\right).$$

Function	Fourier Transform
$f_1(x) = \Pi(x)$	$F_1(s) = \text{sinc}(s)$.

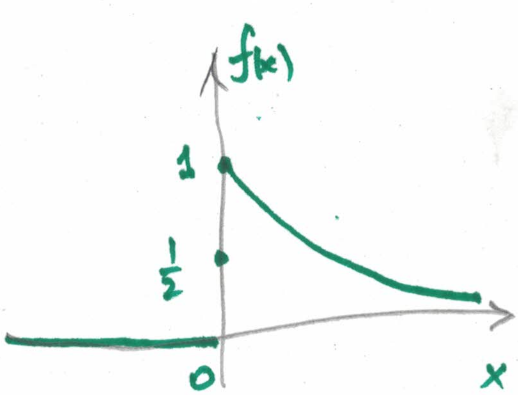
$$f_2(x) = f_1\left(x - \frac{11}{18}\right) \xrightarrow[\tau = \frac{11}{18}]{\text{Translation}} F_2(s) = e^{-2\pi i s \cdot \frac{11}{18}} F_1(s) = e^{-2\pi i \frac{11}{18} s} \cdot \text{sinc}($$

$$f_3(x) = f_2\left(\frac{1}{9}x\right) = \Pi\left(\frac{1}{9}x - \frac{11}{18}\right). \xrightarrow[a = \frac{1}{9}]{\text{Scaling}} F_3(s) = 9F_2(9 \cdot s) = 9e^{-11\pi i s} \cdot \text{sinc}($$

$$f(x) = 3 \cdot f_3(x) \xrightarrow[\alpha = 3,\ \beta = 0]{\text{linearity}} F(s) = 3F_3(s) = 27\, e^{-11\pi i s} \cdot \text{sinc}(9s)$$

Example: The Truncated Decaying Exponential (One-Side Exponential)

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \begin{cases} e^{-x}, & x > 0 \\ \frac{1}{2}, & x = 0 \\ 0, & x < 0 \end{cases}$$



$$F(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} f(x)\, dx = \int_0^{\infty} e^{-2\pi i x s} \cdot e^{-x}\, dx =$$

$$= \int_0^{\infty} e^{-x(2\pi i s + 1)}\, dx = \frac{1}{-(1 + 2\pi i s)} \left(e^{-x(2\pi i s + 1)} \right|_0^{\infty}$$

$$\lim_{x \to \infty} e^{-x(2\pi i s + 1)} = \lim_{x \to \infty} \left(\underbrace{e^{-2\pi i x s}}_{|e^{-2\pi i x s}| = 1} \cdot \underbrace{e^{-x}}_{\to 0} \right) = 0.$$

$$F(s) = \frac{1}{-(1+2\pi i s)} \cdot \Big(\underbrace{\lim_{x\to\infty} e^{-x(2\pi i s+1)}}_{0} - \underbrace{e^{-0\cdot(2\pi i s+1)}}_{1} \Big) =$$

$$= \frac{1}{1+2\pi i s}$$

Recap: $f(x) = \begin{cases} e^{-x}, & x>0 \\ 0, & x<0 \end{cases} \longrightarrow F(s) = \dfrac{1}{1+2\pi i s}$

Rules: Reflection and Conjugation

	function		Fourier Transform
If:	f	$\longrightarrow$	F
Then:	$g(x) = f(-x)$	$\longrightarrow$	$G(s) = F(-s)$.
	$h(x) = \overline{f(x)}$	$\longrightarrow$	$H(s) = \overline{F(-s)}$

Note: $\overline{f(x)}$ denotes complex conjugate of $f(x)$.

Why:

G: follows from Scaling rule with $a = -1$.

$$H(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} h(x)\, dx = \int_{-\infty}^{\infty} e^{-2\pi i x s} \cdot \overline{f(x)}\, dx =$$

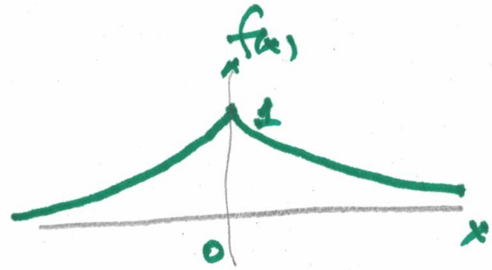
$$= \int_{-\infty}^{\infty} \overline{e^{2\pi i x s} \cdot f(x)}\, dx = \overline{\int_{-\infty}^{\infty} e^{2\pi i x s} f(x)\, dx} = \overline{\int_{-\infty}^{\infty} e^{-2\pi i x(-s)} f(x)\, dx} =$$

$$= \overline{F(-s)} \quad \square$$

Example.

$$f(x) = e^{-|x|}$$

$$F = ?$$



Solution:

If. $h(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x < 0 \end{cases} \longrightarrow H(s) = \dfrac{1}{1+2\pi i s}$.

Then $f(x) = e^{-|x|} = h(x) + h(-x)$

$$\xrightarrow[\text{Reflection Rule}]{\text{by}} F(s) = H(s) + H(-s) = \frac{1}{1+2\pi i s} + \frac{1}{1-2\pi i s} =$$

$$= \frac{1-2\pi i s + 1 + 2\pi i s}{(1+2\pi i s)(1-2\pi i s)} = \underline{\underline{\frac{2}{1+4\pi^2 s^2}}}$$