MATH 632: Functional Analysis

Goal: $\boxed{\text{Functional Calculus.}}$

$T: H \to H$, T linear operator on Hilbert space H.

$f: \mathbb{C} \to \mathbb{C}$, f function.

Want: $f(T)$ --→ Meaning & Properties ...

- Discussions on: Metric Spaces, Normed Vector Spaces, Banach Spaces & Hilbert Spaces. Topological Spaces.

- Baire Category Theorem.
  - → Open Mapping Theorem
  - → closed Graph Theorem
  - → Banach - Steinhaus Theorem.
  - → Hahn - Banach Theorem. --→ Duality in Normed Vector Spaces

    $(V, \|\cdot\|) \hookrightarrow V^{**}$.

- Types of Convergence / Topology:

  weak topology, weak* topology, strong topology, norm topology

- Compact Operators → First View of Functional Calculus

- Algebra of bounded operators $B(H)$.

  subalgebras --→ Banach Algebras. --→ Gelfand Formula $\rho_A(T) = \lim_{n\to\infty} (\|T^n\|_A)^{1/n}$

  --→ Von Neumann Algebras (W^*-algebras).

$T^*T = TT^*$

$T \in$ Banach Algebra

$f : \mathcal{O} \to \mathbb{C}$, Holomorphic.

$\mathcal{O} \subset \mathbb{C}$ open set

$\boxed{\text{Holomorphic Functional Calculus}}$.

$$f(T) = \frac{1}{2\pi i} \oint_{\Gamma} (zI - T)^{-1} \, dz$$

Consequences :

* Wiener Lemma

* Wiener-Levy Theorem.

T : normal operator.

(self-adjoint & unitary)

$f : \mathbb{R} \to \mathbb{C}$, Borel function

$$f(T) = \int_{\sigma(T)} f(\lambda) \underbrace{dP_\lambda}$$

PVM : Projection Valued-Measure

$$T = \int_{\sigma(T)} \lambda \, dP_\lambda.$$

Extensions:

$\boxed{\text{Unbounded Self-Adjoint Operators}}$

(Von Neumann Theory of U.S.A.).