Inverse Mapping Theorem, Lower Bound Theorem, and the Closed Graph Theorem.

Theorem [Inverse Mapping Theorem] A continuous linear bijection of one Banach space onto another has a continuous inverse.

In other words:

If $T : (X, \|\cdot\|_X) \longrightarrow (Y, \|\cdot\|_Y)$ is a linear map such that:

(1) T is bounded.

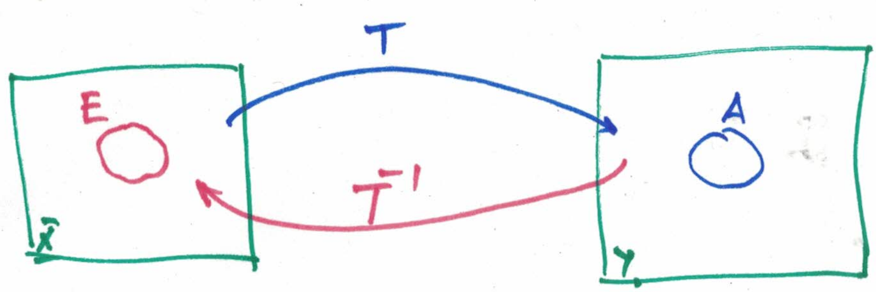
(2) T is invertible.

(3) $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ are Banach spaces

Then

$$T^{-1} : (Y, \|\cdot\|_Y) \longrightarrow (X, \|\cdot\|_X) \text{ is also bounded.}$$

Proof.



By the open mapping theorem: if $E \subset X$ open then $T(E)$ is open in Y.

$\rightarrow$ ~~Pick $A \subset Y$ an open set~~. Pick an arbitrary $E \subset X$ open.

Let $A = T(E)$.

open in Y, $\rightarrow$ ~~$\Rightarrow T^{-1}(A)$ is open in X because T is continuous~~.

$\Rightarrow (T^{-1})^{-1}(E) = T(E) = A$ open in Y $\rightarrow$ T^{-1} is continuous. ∎

Corollary: (Norm Equivalence). Assume $\underline{X}$ is a vector space and $\|\cdot\|_1$, $\|\cdot\|_2$ are two norms on $\underline{X}$ such that $(\underline{X}, \|\cdot\|_1)$ and $(\underline{X}, \|\cdot\|_2)$ are Banach spaces. Assume $i : \underline{X} \to \underline{X}$, $i(x) = x$ satisfies:

there is $B > 0$ s.t. $\|x\|_2 \leq B \cdot \|x\|_1$, $\forall x \in \underline{X}$ ($i : (\underline{X}, \|\cdot\|_1) \to (\underline{X}, \|\cdot\|_2)$ is bounded).

Then there exists $A > 0$ s.t. $\forall x \in \underline{X}$, $\|x\|_2 \geq A \cdot \|x\|_1$.

In other words, $\|\cdot\|_1 \sim \|\cdot\|_2$ (the two norms are equivalent).

Pf.

$i : (\underline{X}, \|\cdot\|_1) \to (\underline{X}, \|\cdot\|_2)$ is bounded, linear, bijection

$\Rightarrow$ $i^{-1} : (\underline{X}, \|\cdot\|_2) \to (\underline{X}, \|\cdot\|_1)$ is also bounded:

$$\exists k : \|x\|_1 \leq k \cdot \|x\|_2, \forall x \Rightarrow \|x\|_2 \geq \frac{1}{k} \|x\|_1$$

$$A \cdot \|x\|_1 \leq \|x\|_2 \leq B \cdot \|x\|_1$$

Remark: $L^2(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} : \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty \}$.

$H^1(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : \int_{-\infty}^{\infty} |f'(x)|^2 dx < \infty \} \subset L^2(\mathbb{R})$.

$\hookrightarrow \|f\|_{H^1} = \|f\|_2 + \|f'\|_2$: $(H^1, \|\cdot\|_{H^1})$ is Banach.

$\forall f \in H^1$: $\|f\|_2 \leq \|f\|_{H^1}$ $(H^1, \|\cdot\|_2)$ is not complete.

BUT: $\exists B > 0$ s.t.
$$\|f\|_{H'} \nleq B \cdot \|f\|_{2}, \quad \forall f \in H'$$

Theorem [Lower Bound Theorem]. Let $T: X \to Y$ be a one-one (injective) bounded linear map between two Banach spaces. Assume $\mathrm{Ran}(T)$ is a closed linear subspace of Y. Then there exists $a > 0$ such that $\|Tx\|_Y \geq a \cdot \|x\|_X$, for all $x \in X$.

Proof

Since $\mathrm{Ran}(T) \subset Y$ is a closed subspace

$$\Rightarrow \left(\mathrm{Ran}(T), \|\cdot\|_Y\right) \text{ is a Banach space.}$$

$T: X \to \mathrm{Ran}(T)$: T is linear, bounded and bijective.

$$\Rightarrow T^{-1}: \mathrm{Ran}(T) \to X \text{ is bounded:}$$

$\exists K > 0$:
$\forall y \in \mathrm{Ran}(T)$: $\|T^{-1}(y)\|_X \leq K \cdot \|y\|_Y$

$$\mathrm{Ran}(T) \ni y = Tx \Rightarrow \|x\|_X \leq K \cdot \|Tx\|_Y, \quad \forall x \in X.$$

$$\Rightarrow \|Tx\|_Y \geq \underbrace{\frac{1}{K}}_{=:a} \|x\|_X .$$

☒

Theorem [Closed Graph Theorem].

Let $T: X \to Y$ be a linear map, not necessarily bounded.

Assume $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces.

If the graph $\Gamma(T) = \{ x \oplus T(x) \} \subset X \oplus Y$ is a closed subspace of $X \oplus Y$ then T is bounded.

Remark: $\Gamma(T)$ is always a linear subspace:

$$\left.\begin{array}{l} (x_1, T(x_1)) \in \Gamma(T) \\ (x_2, T(x_2)) \in \Gamma(T). \\ a_1, a_2 \in \mathbb{C} \end{array}\right| \to \begin{array}{l} a_1(x_1, T(x_1)) + a_2(x_2, T(x_2)) = \\ = (a_1 x_1 + a_2 x_2, a_1 T(x_1) + a_2 T(x_2)) = \\ = (a_1 x_1 + a_2 x_2, T(a_1 x_1 + a_2 x_2)) \in \Gamma(T) \end{array}$$

Proof.

1. $\Gamma(T)$ is closed in $X \oplus Y$ $\Rightarrow$ $(\Gamma(T), \|\cdot\|_\Gamma)$ is a Banach space

$$\|(x, T(x))\|_\Gamma = \|x\|_X + \|Tx\|_Y$$

2. $\Pi_1 : \Gamma(T) \to X$, $\Pi_1(x,y) = x$: $\|\Pi_1(x,y)\|_X \leq \|(x,y)\|_\Gamma$

$\Pi_2 : \Gamma(T) \to Y$, $\Pi_2(x,y) = y$. : $\|\Pi_2(x,y)\|_Y \leq \|(x,y)\|_\Gamma$

$\Rightarrow$ Π_1, Π_2 are linear and bounded maps.

$\Pi_1 : \Gamma(T) \to X$ is invertible.

$$(x, T(x)) \longrightarrow x.$$

$$\Pi_1^{-1} \qquad \Pi^{-1}(x) = (x, T(x)).$$

By inverse mapping theorem: $\Pi_1^{-1} : X \to \Gamma(T)$ is bounded:

$$\exists K: \underbrace{\| \Pi_1^{-1}(x) \|_{\Gamma}}\le K \cdot \|x\|_X$$

$$0 \le \|x\|_X + \|Tx\|_Y \le K \cdot \|x\|_X$$

$$\Rightarrow 0 \le \|Tx\|_Y \le (K-1) \cdot \|x\|_X, \quad \forall x.$$

$\Rightarrow T$ is bounded.

$\blacksquare$

Remark In general.

$T : X \to Y$. What does it mean that T is continuous?

1. Assume $x_n \in X$, $x_n \to x$ in X.
2. $Tx_n \in Y$: i). Tx_n is convergent in Y, say to $y \in Y$.

   ii). $T(x) = y$.

In general you need to show that 1 $\Rightarrow$ (2i & 2ii)

In order to apply the closed graph theorem, you need:

1) $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, $(\mathbb{Y}, \|\cdot\|_{Y})$ are Banach spaces.

2). to show $\Gamma(T)$ is closed:

Take. $(x_n, T(x_n))$ convergent:

Assume 1) $x_n \to x$. in $\mathbb{X}$

2) $Tx_n \to y$. in $\mathbb{Y}$.

Need to check/prove only that $y = T(x)$.

In other words: $\left(1 \text{ and } 2i \right) \Rightarrow 2ii$

<u>Corollary</u> [The Hellinger – Toeplitz Theorem]. Let A be an everywhere defined linear operator on a Hilbert space H such that $\langle Ax, y \rangle = \langle x, Ay \rangle$, for all $x, y \in H$. Then A is bounded.

$A : H \to H$, $(H, \langle \cdot, \cdot \rangle)$ Hilbert space.

[Any symmetric operator defined <u>everywhere</u> on a Hilbert Space must be bounded]

Proof:

graph of A.

Sufficient to show $\Gamma(A)$ is closed.

Take $(x_n, A(x_n))_n$ convergent in $H \oplus H$:

$\Rightarrow$ $x_n \longrightarrow x$.

$A(x_n) \longrightarrow y$.

We need to show $y = A(x)$.

Let $z \in H$:

$$\Big[\langle y, z \rangle = \langle \lim_{n\to\infty} A(x_n), z \rangle =$$

$$= \lim_{n\to\infty} \langle A(x_n), z \rangle = \lim_{n\to\infty} \langle x_n, A(z) \rangle = \langle \lim_{n\to\infty} x_n, A(z) \rangle =$$

$$= \langle x, A(z) \rangle = \underline{\langle A(x), z \rangle}. \quad , \forall z \in H.$$

$\Rightarrow$ $y = A(x)$ $\Rightarrow$ $(x, y) \in \Gamma(A)$

By closed graph theorem $\Rightarrow$ A is bounded.

$\square$

<u>Theorem</u>. For any $x_0 \in [0,1)$ there is $\underline{f \in C[0,1]}$, $f(0)=f(1)$ s.t. $\left(S_n f(x_0)\right)_{n \geq 1}$ is unbounded, hence not convergent to $f(x_0)$

where $(S_n f)(x) = \sum_{k=-n}^{n} c_k e^{2\pi i k x}$, $\quad c_k = \int_0^1 e^{-2\pi i k x} f(x)\,dx$.

Pf. Using Uniform boundedness principle:

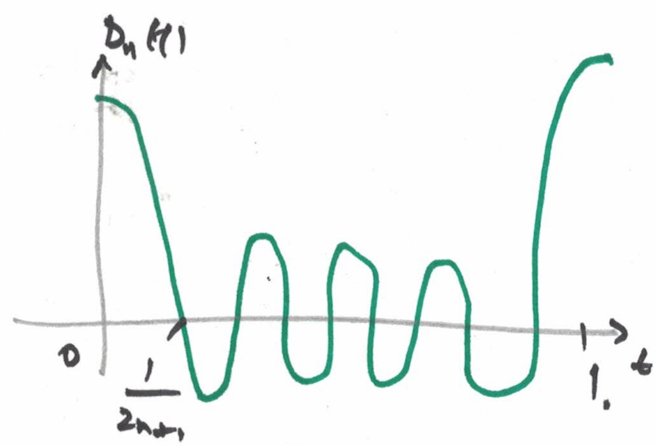
$$T_n : f \longmapsto (S_n f)(x_0).$$

1. $$T_n(f) = \sum_{k=-n}^{n} c_k e^{2\pi i k x} = \int_0^1 \underbrace{\sum_{k=-n}^{n} e^{2\pi i k(x-y)}}_{D_n(x-y)} f(y)\,dy = \int_0^1 D_n(x-y) f(y)\,dy.$$

$$D_n(t) = \frac{\sin(\pi(2n+1)t)}{\sin(\pi t)}$$

↓

Dirichlet kernel.



2. $$\|T_n\|_{C[0,1]^*} = \int_0^1 |D_n(t)|\,dt = \|D_n\|_1$$

3. Compute: $\|D_n\|_1 \sim c \cdot \log(n)$.

Assume. $S_n f(x_0) = T_n f$ is ~~convergent~~ bounded. for every $f \in C[0,1]$.

$$\Longrightarrow \{T_n f\}_{n \geq 1} \text{ is bounded. } \forall f.$$

1)

2). each T_n is bounded

$$T_n : C[0,1] \to \mathbb{C}.$$

$\Longrightarrow$ By uniform boundedness principle $\Rightarrow$ $\{\|T_n\|\}$ must be bounded.

But $\|T_n\| = \|D_n\|_1 \sim c \cdot \log(n) \nearrow \infty$

$\Rightarrow$ contradiction. $\boxtimes$

(Du Bois-Raymond proof).