L12

Compact Sets in Banach Spaces

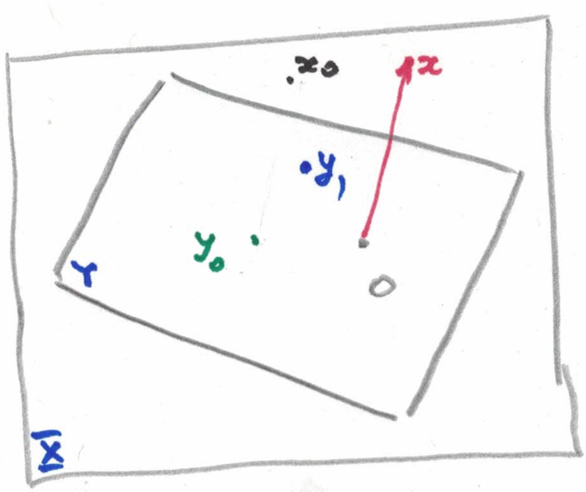
Goal: If the closed unit ball is compact then the space must be finite dimensional.

Setup: Assume $(\underline{X}, \|\cdot\|)$ is a Banach space.

Lemma [Riesz Geometric Lemma]. Let Y be a closed proper subspace of a normed linear space, $(\underline{X}, \|\cdot\|)$. Then for any $\varepsilon > 0$, there exist $x \in \underline{X}$ such that $\|x\| = 1$, $\operatorname{dist}(x, Y) \geq 1 - \varepsilon$.

$$\left(\underbrace{\operatorname{dist}(x, Y)}_{d_Y(x)} = \inf_{y \in Y} \|x - y\| . \right).$$

Proof



Pick $x_0 \in \underline{X} \setminus Y$

Let $d_0 = d_Y(x_0) = \inf_{y \in Y} \|x_0 - y\| > 0$.

Let $y_0 \in Y$:

$$\|x_0 - y_0\| \leq \frac{1}{1-\varepsilon} \cdot d_0 = \frac{\inf_{y \in Y} \|x_0 - y\|}{1 - \varepsilon}.$$

Set $x = \dfrac{x_0 - y_0}{\|x_0 - y_0\|}$.

Let $\underline{y_1 \in Y}$. Want: $\|x - y_1\| \geq 1 - \varepsilon$. $\dashrightarrow$ $d_Y(x) = \inf_{y \in Y} \|x - y\| \geq 1 - \varepsilon$

$\longrightarrow$ construct $y_2 = y_0 + \|x_0 - y_0\| \cdot y_1 \in Y$.

$$\|x-y_1\| = \left\| \frac{x_0-y_0}{\|x_0-y_0\|} - \frac{y_2-y_0}{\|x_0-y_0\|} \right\| = \frac{\|x_0-y_2\|}{\|x_0-y_0\|} \geq \frac{d_0}{\frac{1}{1-\varepsilon} d_0} = 1-\varepsilon$$

$$\Downarrow$$

$$d_Y(x) \geq 1-\varepsilon.$$

<u>Theorem</u>. Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Suppose.

$\overline{B}_1 = \{x \in \mathfrak{X} : \|x\| \leq 1\}$ is compact. Then $\dim \mathfrak{X} < \infty$.

<u>Proof</u>.

<u>Recall</u>: A set $S \subset \mathfrak{X}$ is compact iff. from any sequence $(x_n)_{n\geq 1}$ we can extract a convergent subsequence.

Assume $\dim \mathfrak{X} = +\infty$. Then construct inductively:

$x_1 \in \mathfrak{X}$, $\|x_1\| = 1$.

$x_2 \in \mathfrak{X}$, $\|x_2\| = 1$ s.t. $\|x_1 - x_2\| \geq \frac{1}{2}$ ---> use Riesz geometric lemma

$x_3 \in \mathfrak{X}$, $\|x_3\| = 1$ s.t. $d(x_3, \operatorname{span}(x_1, x_2)) \geq \frac{1}{2}$ ---> $\|x_1 - x_3\| \geq \frac{1}{2}$

$\|x_2 - x_3\| \geq \frac{1}{2}$

$\vdots$

$x_n \in \mathfrak{X}$, $\|x_n\| = 1$ ..

$x_{n+1} \in \mathfrak{X}$, $\|x_{n+1}\| = 1$ s.t. $\operatorname{dist}(x_{n+1}, \operatorname{span}(x_1, \ldots x_n)) \geq \frac{1}{2}$

$\vdots$

$\longrightarrow \|x_{n+1} - x_k\| \geq \frac{1}{2}, \ \forall \ k \leq$

Thus $(x_n)_{n\geq 1} \subset \underline{X}$: $\|x_n\| = 1 \longrightarrow x_n \in \overline{B_1}$.

and $\underline{\|x_n - x_m\| \geq \frac{1}{2}}$, $n \neq m$. $\dashrightarrow$ $(x_n)_n$ & any subsequence is not Cauchy

If $\overline{B_1}$ is compact $\dashrightarrow \exists\, (x_{n_k})_k$ convergent. $\dashrightarrow$ Cauchy. $\longrightarrow$

$\longrightarrow$ contradiction. $\blacksquare$

<u>Conclusion:</u>

<u>Theorem.</u> Let $(\underline{X}, \|\cdot\|)$ be a normed linear space.
The closed unit ball $\overline{B_1}$ is compact w.r.t. norm topology if and only if $\dim \underline{X} < \infty$.

Assume $\underline{X}$ is a Banach space and let $B(\underline{X}) = B(\underline{X}, \underline{X})$ denote the set of bounded linear operators from $\underline{X}$ to $\underline{X}$. (on $\underline{X}$).

If $T, S \in B(\underline{X})$ then $T \cdot S \in B(\underline{X})$, and:

$$\|T\cdot S\|_{B(\underline{X})} \leq \|T\|_{B(\underline{X})} \cdot \|S\|_{B(\underline{X})}.$$

<u>Consequence:</u> $\left(B(\underline{X}), +, \overset{\text{composition,}}{\cdot}\right)$ is a Banach algebra.

1). $(B(\underline{X}), +)$ is a Banach space.

2). $(B(\underline{X}), +, \cdot)$ is an algebra.

3) $\|T\cdot S\| \leq \|T\| \cdot \|S\|$.

Furthermore: $(B(\underline{X}), +, \cdot)$ is a <u>unital</u> Banach algebra.

$\mathbb{1} \in B(\underline{X})$.
$\mathbb{1}(x) = x$

Let $T \in B(\underline{X})$.

<u>Definition</u> The **resolvent** of T is the set $\rho = \rho(T) = \rho_T \subset \mathbb{C}$ of complex numbers z such that $z \cdot 1 - T$ is invertible.

$$(z \cdot 1 - T)^{-1} \in B(\underline{X}) \iff \exists S \in B(\underline{X}) \text{ s.t. } (zI - T) \cdot S = S \cdot (zI - T) = I$$

$$\rho(T) = \left\{ z \in \mathbb{C} : z \cdot I - T \text{ is invertible} \right\}.$$

<u>Definition</u>. The **spectrum** of T denoted $\sigma = \sigma(T) = \sigma_T \subset \mathbb{C}$ is the complement of the resolvent, $\sigma(T) = \mathbb{C} \setminus \rho(T)$:

$$\sigma(T) = \left\{ z \in \mathbb{C} : z \cdot I - T \text{ is \underline{not} invertible} \right\}.$$

<u>Remark</u>.

If $z \in \sigma(T)$, then (a) $z \cdot I - T$ is not surjective (onto).
or
(b) ~~$z \cdot I$ Ker~~ $zI - T$ is not injective (one-one)

$\iff$ (a) $\mathrm{Ran}(z \cdot I - T) \neq \underline{X}$.

(b) $\ker(z \cdot I - T) \neq \{0\}$.

<u>Definition</u>. A complex number $z \in \sigma(T)$ is called an **eigenvalue** if. $\ker(zI - T) \neq \{0\}$.

If z is an eigenvalue for T, then any non-zero vector in $\ker(zI - T)$ is called an eigenvector.

$\lambda \in \sigma(T)$ eigenvalue $\longrightarrow \exists\, x \neq 0$ s.t. $Tx = \lambda x$.

such x is an eigenvector.

The set of eigenvalues is called the pure point spectrum, denoted $\sigma_{pp.}(T)$.

Remark. Assume. $(A, +, \cdot, \|\cdot\|)_{\mathbb{C}}$ a normed unital algebra.

Let $x \in A$. The spectrum of x with respect to A, $\sigma_A(x)$,

$$\sigma_A(x) = \{ z \in \mathbb{C} : z \cdot 1 - x \text{ is not invertible in } A \}.$$

Resolvent:
$$\rho_A(x) = \{ z \in \mathbb{C} : \exists\, y \in A \text{ s.t. } (z \cdot 1 - x) y = y (z \cdot 1 - x) = 1 \}.$$

$$\sigma_A(x) = \mathbb{C} \setminus \rho_A(x).$$

Example. $B(C[0,1]) = \{ T : C[0,1] \longrightarrow C[0,1],\ T \text{ bounded} \}$.

$$A = \{ f = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x},\ \sum_n |a_n| < \infty \}.$$ $\longrightarrow$ collection of absolutely summable Fourier series.

$$f \in A \dashrightarrow M_f : C[0,1] \to C[0,1].$$
$$g \longmapsto f \cdot g$$

(pointwise multiplication)

$$M_f \in B(C[0,1]).\ ,\ M_f \in B(A).$$

$$M_A = \{ M_f : f \in A \} \subset B(C[0,1]).$$

$\hookrightarrow$ subalgebra of

$\left(M_A, \|\cdot\|_{B(A)}\right) \subset B(C[0,1], C[0,1])$

$\left(M_A, \|\cdot\|_{B(C[0,1])}\right) \subset B(C[0,1], C[0,1])$

Return to $B(\underline{X})$, $\underline{X}$: Banach space.

Definition. An operator $T \in B(\underline{X})$ is said to have the **finite approximation property** if for any $\varepsilon > 0$ there exists a finite rank operator $T_\varepsilon \in B(\underline{X})$ s.t. $\|T - T_\varepsilon\|_{B(\underline{X})} < \varepsilon$.

$\updownarrow$

$\dim \operatorname{Ran}(T_\varepsilon) < \infty$

Definition An operator, $T \in B(\underline{X})$ is said **compact**

if. $\overline{T(\overline{B}_1)}$ is compact.

"The image of the closed unit ball is pre-compact."

$A \subset \underline{X}$ is called pre-compact if $\overline{A}$ is compact

Definition. An operator $T \in B(\underline{X})$ is said **completely continuous**

if. $\forall\, (x_n)_{n\geq 1}$, weakly convergent sequence in $\underline{X}$, $(T(x_n))_{n\geq 1}$ is convergent in $\underline{X}$.

$$\text{If} \quad x_n \xrightarrow{w} x \quad \Longrightarrow \quad Tx_n \longrightarrow Tx.$$

$$\Updownarrow$$

$$\left[\forall\, \ell \in \underline{X}^*, \lim_{n\to\infty} |\ell(x_n - x)| = 0\right] \Rightarrow \lim_{n\to\infty} \|T(x_n - x)\| = 0.$$

Remark. For general Banach spaces:

$$\underbrace{FA(\underline{X})}_{\text{set of operators that have finite approx. property}} \subset \underbrace{C(\underline{X})}_{\text{set of compact operators on } \underline{X}} \subset \underbrace{CC(\underline{X})}_{\text{set of completely continuous operators}}.$$

Remark. If $(\underline{X}, \|\cdot\|)$ is reflexive, $\underline{X}^*$ separable, $\underline{X}$ has a Schauder basis

then: $FA(\underline{X}) = C(\underline{X}) = CC(\underline{X})$.

If $X = H$ is a Hilbert space then:

$$FA(H) = C(H) = CC(H).$$

<u>Example</u>: let $X = C[0,1]$, $\|\cdot\|_\infty = \|\cdot\|$.

let $k : [0,1]\times[0,1] \to \mathbb{C}$, be a continuous function $\Big\} \to$ kernel of operator T

$$k \in C([0,1]\times[0,1]).$$

let $T : C[0,1] \longrightarrow C[0,1]$, $\quad Tf(x) = \int_0^1 K(x,y)\, f(y)\,dy.$

<u>Claim</u>: T is a compact operator:

1). $$|Tf(x)| = \left|\int_0^1 K(x,y)\, f(y)\, dy\right| \le \int_0^1 |K(x,y)|\cdot|f(y)|\,dy \le$$

$$\le \sup_{y\in(0,1)} |f(y)| \cdot \int_0^1 |K(x,y)|\,dy \le \|K\|_\infty \cdot \|f\|_\infty .$$

$$\Rightarrow \quad \|T\|_{B(C[0,1])} \le \|K\|_\infty . \qquad \longrightarrow T \text{ bounded.}$$

2) let $f \in \overline{B_1(0)}$: $\|f\|_\infty \le 1$.

$$|Tf(x_1) - Tf(x_2)| = \left|\int_0^1 (K(x_1,y) - K(x_2,y))\, f(y)\,dy\right| \le$$

$$\le \int_0^1 |K(x_1,y) - K(x_2,y)|\,dy.$$

K continuous. $[0,1]\times[0,1]$ compact $\Rightarrow$ $\underline{K}$ is uniformly continu..

$\Rightarrow$ For any $\varepsilon > 0$ $\exists$ δ s.t. $\underline{|x_1 - x_2| < \delta_\varepsilon} \Rightarrow |K(x_1, y) - K(x_2, y)| < \varepsilon$

$\Rightarrow$ $|Tf(x_1) - Tf(x_2)| \leq \underline{\varepsilon}$.

$\overline{B_1(0)} \xrightarrow{T} \{ Tf : f \in \overline{B_1(0)} \}$ is $\underline{\text{equi}}$continuous.

$\delta = \delta(\varepsilon)$ independent of f.

& bounded in $C[0,1]$.

$\Longrightarrow$ By Arzela-Ascoli : From any f_n : $\{ (Tf_n)_{n \geq 1}$ is equicont & bounded
$\|f_n\| \leq 1$

$\longrightarrow$ $\exists$ convergent subsequence w.r.t. $\|\cdot\|_\infty$ norm

$\longrightarrow$ $\overline{T(\overline{B_1(0)})}$ is compact.

$\Longrightarrow$ T is $\underline{\text{compact}}$.