Banach Algebras

Definition Let $\mathcal{U}$ be a Banach algebra with identity.
Let $x \in \mathcal{U}$. Then the set

$$\rho_{\mathcal{U}}(x) = \{\lambda \in \mathbb{C} \mid \lambda e - x \text{ has a two-sided inverse}\}$$

is called the resolvent set of x with respect to $\mathcal{U}$.

The set $\sigma_{\mathcal{U}}(x) = \mathbb{C} \setminus \rho_{\mathcal{U}}(x)$ is called the spectrum of x w.r.t. $\mathcal{U}$.

$$\sigma_{\mathcal{U}}(x) = \{\lambda \in \mathbb{C} : \lambda \cdot e - x \text{ is not invertible}\}.$$

Remark:

x has two-sided inverse $\Leftrightarrow$ x is invertible

$$\exists\, y, z \in \mathcal{U} : x \cdot y = z \cdot x = e. \Leftrightarrow \left[\exists\, y \in \mathcal{U} : x \cdot y = y \cdot x = e.\right]$$

$$(z = z \cdot e = z \cdot x \cdot y = e \cdot y = y.)$$

Notation: From now on, the identity element e will be denoted by 1.

Proposition. Suppose $\mathcal{U}$ is a Banach algebra with identity and $x \in \mathcal{U}$ has a two-sided inverse (i.e., invertible). If $y \in \mathcal{U}$,

$\|y\| < \dfrac{1}{\|x^{-1}\|}$ then $x + y$ is invertible in $\mathcal{U}$ and:

$$(x+y)^{-1} = \sum_{n=0}^{\infty} x^{-1} \cdot (-y \cdot x^{-1})^n$$

In particular, if $x = 1$ and $\|y\| < 1$ then:

$$(1+y)^{-1} = \sum_{n=0}^{\infty} (-1)^n y^n = 1 - y + y^2 - y^3 + \cdots, \quad (1-y)^{-1} = \sum y^n = 1 + y + y^2 + \cdots.$$

Sketch of Proof:

$$\|y\| < \frac{1}{\|x^{-1}\|} \Rightarrow \|-y\cdot x^{-1}\| \le \|y\|\cdot\|x^{-1}\| < 1.$$

$\Rightarrow$ series. $\sum\limits_{n\ge 0} (-y\cdot x^{-1})^n$. is convergent in $\mathcal{U}$ (w.r.t. $\mathcal{U}$-norm)

Let $z = \sum\limits_{n=0}^{\infty} x^{-1}\cdot(-y\cdot x^{-1})^n$.

$$x\cdot z = \sum_{n=0}^{\infty}(-y\cdot x^{-1})^n = 1 - y\cdot x^{-1} + y x^{-1} y x^{-1} - y x^{-1} y x^{-1} y x^{-1} + \cdots$$

$$= 1 - y\cdot\left[x^{-1} - x^{-1} y x^{-1} + x^{-1} y x^{-1}\cdot y x^{-1} - \cdots\right] =$$

$$= 1 - y\cdot\sum_{n=0}^{\infty} x^{-1}\cdot(-y x^{-1})^n = 1 - y\cdot z.$$

$$\Rightarrow (x+y)\cdot z = 1.$$

Similarly: $z\cdot x = 1 - z\cdot y$.

(Remark) Corollary: If. $y \in \mathcal{U}$, $\|y\|\cdot\|x^{-1}\| \le r < 1$ then:

$$\|(x+y)^{-1}\| = \left\|x^{-1}\cdot\sum_{n=0}^{\infty}(-y x^{-1})^n\right\| \le \|x^{-1}\|\cdot\sum_{n=0}^{\infty}\left(\|y\|\cdot\|x^{-1}\|\right)^n =$$

$$= \frac{\|x^{-1}\|}{1 - \|y\|\cdot\|x^{-1}\|} \le \frac{\|x^{-1}\|}{1-r}.$$

Corollary Let $\mathcal{U}$ be a Banach algebra with identity.
The set of invertible element is open, and the set of non-invertible elements is closed.

Pf:
Let $x \in \mathcal{U}$ invertible. Then $B_r(x) = \{ y \in \mathcal{U} : \|x-y\| < r \}$ is open:

where $r = \dfrac{1}{\|x^{-1}\|}$

If. $y \in B_r(x) \to y = x + w_x$ with $\|w\| < r = \dfrac{1}{\|x^{-1}\|}$.

by Proposition $\Rightarrow$ $x + w$ is invertible $\Rightarrow$ y is invertible.

$\boxtimes$

Theorem. Let $\mathcal{U}$ be a Banach algebra with identity.
Then for any $x \in \mathcal{U}$:

resolvent set w.r.t. $\mathcal{U}$ — spectrum w.r.t. $\mathcal{U}$.

(1) $\rho(x)$ is an open set in $\mathbb{C}$; $\sigma(x)$ is a closed set in $\mathbb{C}$.

(2) $R_x : \rho(x) \to \mathcal{U}$, $R_x(\lambda) = (\lambda \cdot 1 - x)^{-1}$ is an analytic function in λ (i.e., holomorphic).

(3) $\sigma(x)$ is not empty.

(4) $\sigma(x) \subset U_{\|x\|} = \{ \lambda \in \mathbb{C} : |\lambda| \leq \|x\| \} = \overline{B}_{\|x\|}(0)$

closed disk of radius $\|x\|$ and center 0 in $\mathbb{C}$

Proof.

(1).

Assume. $\lambda \in \rho(x)$: $\lambda \cdot 1 - x$ is invertible in $\mathcal{U}$.

let $r = \dfrac{1}{\|(\lambda \cdot 1 - x)^{-1}\|}$.

Take $\mu \in B_r(\lambda) \subset \mathbb{C}$: $|\mu - \lambda| < r$.

Consider. $w = \mu \cdot 1 - x \in \mathcal{U}$.

let $v = \lambda \cdot 1 - x \in \mathcal{U}$ and invertible.

$$\|w - v\| = \|(\mu - \lambda) \cdot 1\| = |\mu - \lambda| < r = \frac{1}{\|v^{-1}\|}$$

$w = v + y$, $\|y\| < \dfrac{1}{\|v^{-1}\|}$. By Proposition $\Rightarrow$ w is invertible.

$\Rightarrow \mu \in \rho(x)$,

Thus : $B_r(\lambda) \subset \rho(x) \Rightarrow \rho(x)$ is open.

$\sigma(x) = \mathbb{C} \setminus \rho(x)$ is closed.

(2).

$$\lambda \in \rho(x) \longmapsto R_x(\lambda) = (\lambda \cdot 1 - x)^{-1}.$$

Definition. A function $F : D \to \mathcal{U}$ is called. strongly analytic at $x_0 \in D$, where $D \subset \mathbb{C}$ is an open connected set. if. the limit. $\lim\limits_{h \to 0} \frac{1}{h}\left(F(x_0 + h) - F(x_0)\right)$ exists in $\mathcal{U}$ as $h \to 0$ in $\mathbb{C}$.

Fact: If $F : D \to \mathcal{U}$ is strongly analytic at x_0 then there are $(T_n)_{n\geq 0}$, $T_n \in \mathcal{U}$ ~~s.t.~~ and $r>0$, s.t.

$$F(z) = \sum_{n\geq 0} (z-x_0)^n \cdot T_n.$$

converges in $\mathcal{U}$, absolutely ~~for~~ for every $z \in B_r(x_0) \subseteq \mathbb{C}$.

The radius of convergence: $r(F; x_0) = \dfrac{1}{\limsup_{n\to\infty} \|T_n\|^{1/n}}$.

Lemma. [First Resolvent Formula].

For any $\mu, \lambda \in \rho(x)$, $R_x(\lambda) \cdot R_x(\mu) = R_x(\mu) \cdot R_x(\lambda)$ (they commute), and:

$$R_x(\lambda) - R_x(\mu) = -(\lambda - \mu) \cdot R_x(\lambda) \cdot R_x(\mu).$$

Proof.

(1). $(\lambda \cdot 1 - x) \cdot (\mu \cdot 1 - x) = \lambda\mu 1 - \lambda x - \mu x + x^2$

$(\mu \cdot 1 - x)(\lambda \cdot 1 - x) = \lambda\mu 1 - \mu x - \lambda x + x^2$

$\rightarrow$ use: $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$

$\Rightarrow R_x(\lambda) \cdot R_x(\mu) = R_x(\mu) \cdot R_x(\lambda)$.

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(2) $R_x(\lambda) - R_x(\mu) = (\lambda \cdot I - x)^{-1} - (\mu \cdot 1 - x)^{-1} = (\lambda \cdot 1 - x)^{-1} \cdot \left[\mu \cdot 1 - x - (\lambda \cdot 1 - x)\right](\mu \cdot 1 - x)^{-1}$

$= (\mu - \lambda) \cdot (\lambda \cdot 1 - x)^{-1} (\mu \cdot 1 - x)^{-1}$ ...

Return to the proof that $z \longmapsto R_x(z)$ is analytic.

Fix $\lambda \in \rho(x)$. Want: $\displaystyle\lim_{\mu \to \lambda} \frac{1}{\mu-\lambda}\left[R_x(\mu) - R_x(\lambda)\right]$.

exists in $\mathcal{U}$.

By Lemma:

$$\frac{1}{\mu-\lambda}\left[R_x(\mu) - R_x(\lambda)\right] = -R_x(\lambda)\cdot R_x(\mu) =$$

$$= -R_x(\lambda)\cdot\left[R_x(\lambda) + R_x(\mu) - R_x(\lambda)\right] =$$

$$= -(R_x(\lambda))^2 + \underbrace{R_x(\lambda)\cdot R_x(\mu)\cdot R_x(\lambda)}_{\text{need to check if is uniformly bounded for every } \mu \in B_r(\lambda) \text{ for some } r>0}\cdot(\mu-\lambda).$$

Once we have uniform boundedness,

$$\lim_{\mu\to\lambda} \frac{1}{\mu-\lambda}\left[R_x(\mu) - R_x(\lambda)\right] = -\underline{(R_x(\lambda))^2}.$$

Let $r < \dfrac{1}{\|(\lambda\cdot 1 - x)^{-1}\|}$. Let $\mu \in \mathbb{C} : |\mu - \lambda| < r$.

Then: $(\mu\cdot 1 - x)^{-1} = (\underbrace{(\mu-\lambda)\cdot 1}_{\text{perturbation}} + \underbrace{\lambda\cdot 1 - x}_{\text{invertible}})^{-1} =: R_x(\mu)$.

$$\|R_x(\mu)\| \le \frac{\|R_x(\lambda)\|}{1 - |\mu-\lambda|\cdot\|R_x(\lambda)\|} = \frac{\|R_x(\lambda)\|}{1 - r\cdot\|R_x(\lambda)\|}$$

$\forall \mu \in B_r(\lambda)$.

$$\Rightarrow \|R_x(\lambda)\cdot R_x(\mu)\, R_x(\lambda)\| \le \frac{\|R_x(\lambda)\|^3}{1 - r\cdot\|R_x(\lambda)\|}, \forall \mu \in B_r(\lambda)$$

(3) $\sigma(x)$ is not empty.

Assume $\sigma(x)$ is empty $\Rightarrow$ $z\cdot 1 - x$ is invertible for every $z \in \mathbb{C}$.

$z \mapsto R_x(z) = (z\cdot 1 - x)^{-1}$ is holomorphic in the entire complex plane.

Take $l \in \mathcal{U}^*$.

$z \mapsto l(R_x(z)) \in \mathbb{C}$, holomorphic in the entire $\mathbb{C}$.

$$(z\cdot 1 - x)^{-1} = \frac{1}{z}\cdot\left(1 - \frac{1}{z}x\right)^{-1}$$

If. $|z| > \|x\|$ $\Rightarrow$ $\frac{1}{z}X$ is invertible

$$\left\|\left(1 - \frac{1}{z}x\right)^{-1}\right\| \leq \frac{1}{1 - \frac{\|x\|}{|z|}}$$

$$\Rightarrow \|(z\cdot 1 - x)^{-1}\| \leq \frac{1}{|z|}\cdot\frac{1}{1 - \frac{\|x\|}{|z|}} = \frac{1}{|z| - \|x\|}$$

$$\Rightarrow \lim_{z\to\infty} \|(z\cdot 1 - x)^{-1}\| = 0.$$

$\Rightarrow$ $z \mapsto l(R_x(z))$ is bounded.

and $\lim_{z\to\infty} l(R_x(z)) = 0$

By the Liouville's theorem:

$$l(R_x(z)) = 0, \forall z, \forall l \in \mathcal{U}^*$$

$\Rightarrow$ $R_x(z) = 0, \forall z$ $:\Rightarrow$ $z\cdot R_x(z) = 0$

$$0 = \underbrace{z \cdot R_x(z)}_{\text{at } z \to \infty} = z \cdot (z \cdot 1 - x)^{-1} = \underbrace{\left(1 - \frac{1}{z} x\right)^{-1}}$$

$$\Downarrow$$

$$0 \neq 1. \Rightarrow \underline{\underline{\text{Contradiction.}}}$$

(4).

If $\lambda \in \sigma(x)$: need to show : $|\lambda| \leq \|x\|$.

Assume : $\mu \in \mathbb{C}$: $|\mu| > \|x\|$

we shown : $(\mu \cdot 1 - x)^{-1} = \frac{1}{\mu} \cdot \underbrace{\left(1 - \frac{1}{\mu} x\right)^{-1}}_{\in \mathcal{U}} \in \mathcal{U}$.

$\Rightarrow \mu \in \rho(x)$.

$$\Downarrow$$

If $\lambda \in \sigma(x) \Rightarrow |\lambda| \leq \|x\|$.

exists.

$$\forall \ell \in \mathcal{U}^* \to \ell\left(\lim_{\mu \to \lambda} \frac{1}{\mu - \lambda} [R_x(\mu) - R_x(\lambda)]\right) = \Downarrow$$

$$= \lim_{\mu \to \lambda} \frac{1}{\mu - \lambda} [\ell(R_x(\mu)) - \ell(R_x(\lambda))]. \text{ ~~exists~~.}$$